

***Fundamentals of Differential Equations 9th edition
Nagle Solutions Manual***

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CHAPTER 2: First Order Differential Equations

EXERCISES 2.2: Separable Equations

2. This equation is separable because we can separate variables by multiplying both sides by dx and dividing by $4y^2 - 3y + 1$.

4. This equation is separable because

$$\frac{dy}{dx} = \frac{ye^{x+y}}{x^2 + 2} = \left(\frac{e^x}{x^2 + 2} \right) ye^y = g(x)p(y).$$

6. Writing the equation in the form

$$\frac{ds}{dt} = \frac{s+1}{st} - s^2,$$

we see that the right-hand side cannot be represented in the form $g(t)p(s)$. Therefore, the equation is not separable.

8. To separate variables, we divide the equation by x and multiply by dt . Integrating yields

$$\begin{aligned} \frac{dx}{x} = 3t^2 dt &\Rightarrow \ln|x| = t^3 + C_1 &\Rightarrow |x| = e^{t^3+C_1} = e^{C_1}e^{t^3} \\ \Rightarrow |x| = C_2 e^{t^3} &\Rightarrow x = \pm C_2 e^{t^3} = C e^{t^3}, \end{aligned}$$

where C_1 is an arbitrary constant and, therefore, $C_2 := e^{C_1}$ is an arbitrary *positive* constant, $C = \pm C_2$ is any *nonzero* constant. Separating variables, we lost a solution $x \equiv 0$, which can be included in the above formula by taking $C = 0$. Thus, $x = Ce^{t^3}$, C – arbitrary constant, is a general solution.

10. Separating variables yields

$$y^2 dy = \frac{xdx}{\sqrt{1+x}} \Rightarrow \int y^2 dy = \int \frac{xdx}{\sqrt{1+x}}.$$

The second integral can be evaluated using a substitution $u = 1 + x$.

$$\int \frac{xdx}{\sqrt{1+x}} = \int \frac{(u-1)du}{\sqrt{u}} = \int (u^{1/2} - u^{-1/2}) du = \frac{2}{3}u^{3/2} - 2u^{1/2} + C.$$

Thus,

$$\frac{y^3}{3} = \frac{2}{3}(1+x)^{3/2} - 2(1+x)^{1/2} + C \quad \Rightarrow \quad y = \sqrt[3]{2(1+x)^{3/2} - 6(1+x)^{1/2} + C}.$$

12. Separating variables, we obtain

$$\frac{dy}{\sec^2 y} = \frac{dx}{1+x^2}.$$

Using the trigonometric identities $\sec y = 1/\cos y$ and $\cos^2 y = (1 + \cos 2y)/2$ and integrating, we get

$$\begin{aligned} \frac{dy}{\sec^2 y} &= \frac{dx}{1+x^2} \quad \Rightarrow \quad \frac{(1 + \cos 2y)dy}{2} = \frac{dx}{1+x^2} \\ \Rightarrow \quad \int \frac{(1 + \cos 2y)dy}{2} &= \int \frac{dx}{1+x^2} \\ \Rightarrow \quad \frac{1}{2} \left(y + \frac{1}{2} \sin 2y \right) &= \arctan x + C_1 \\ \Rightarrow \quad 2y + \sin 2y &= 4 \arctan x + 4C_1 \quad \Rightarrow \quad 2y + \sin 2y = 4 \arctan x + C. \end{aligned}$$

The last equation defines implicit solutions to the given differential equation.

14. Writing the given equation in the form $dx/dt = x^3 + x$, we separate the variables to get

$$\frac{dx}{x^3 + x} = \frac{dx}{x(x^2 + 1)} = \int \left(\frac{1}{x^2} - \frac{1}{x^2 + 1} \right) x dx = dt.$$

Thus,

$$\frac{1}{2} [\ln(x^2) - \ln(x^2 + 1)] = t + C_1 \quad \Rightarrow \quad \frac{x^2}{x^2 + 1} = e^{2C_1} e^{2t} = C e^{2t},$$

where $C > 0$ is an arbitrary constant. (Note that, separating variables, we lost a solution $x \equiv 0$, which can be included in the above formula by letting $C = 0$.) Solving for x , we get

$$x = \pm \sqrt{\frac{C e^{2t}}{1 - C e^{2t}}}, \quad C \geq 0.$$

16. To separate variables, we move the term, containing dx , to the right-hand side of the equation and divide both sides by y . This yields

$$y^{-1} dy = -y e^{\cos x} \sin x dx \quad \Rightarrow \quad y^{-2} dy = -e^{\cos x} \sin x dx.$$

Integrating the last equation, we obtain

$$\int y^{-2} dy = \int (-e^{\cos x} \sin x) dx \quad \Rightarrow \quad -y^{-1} + C = \int e^u du \quad (u = \cos x)$$

$$\Rightarrow \quad -\frac{1}{y} + C = e^u = e^{\cos x} \quad \Rightarrow \quad y = \frac{1}{C - e^{\cos x}},$$

where C is an arbitrary constant.

18. First we find a general solution to the equation. Separating variables and integrating, we get

$$\begin{aligned} \frac{dy}{dx} &= x^3(1-y) & \Rightarrow & \quad \frac{dy}{1-y} = x^3 dx \\ \Rightarrow & \quad \int \frac{dy}{1-y} = \int x^3 dx & \Rightarrow & \quad -\ln|1-y| + C_1 = \frac{x^4}{4} \\ \Rightarrow & \quad |1-y| = \exp\left(C_1 - \frac{x^4}{4}\right) = Ce^{-x^4/4}. \end{aligned}$$

We use the initial condition, $y(0) = 3$, to find C . Thus, substitution $y = 3$ and $x = 0$ into the last equation yields

$$|1-3| = Ce^{-0^4/4} \quad \Rightarrow \quad 2 = C.$$

Therefore, $|1-y| = 2e^{-x^4/4}$. Finally, since $1-y(0) = 1-3 < 0$, on an interval containing $x = 0$ one has $1-y(x) < 0$ so that $|1-y(x)| = y(x) - 1$. Thus, the answer

$$y - 1 = 2e^{-x^4/4} \quad \text{or} \quad y = 2e^{-x^4/4} + 1.$$

20. Separating variables yields

$$(y+1)dy = \frac{4x^2 - x - 2}{x^2(x+1)} dx. \quad (2.1)$$

For integrating the right-hand side, we use partial fractions.

$$\frac{4x^2 - x - 2}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

Solving for A , B , and C , we get $A = 1$, $B = -2$, and $C = 3$. Thus, integrating (2.1), we obtain

$$\frac{1}{2}(y+1)^2 = \ln x + \frac{2}{x} + 3\ln(x+1) + C.$$

With $y(1) = 1$, $C = -3\ln 2$ so that the answer is

$$\frac{1}{2}(y+1)^2 = \ln x + \frac{2}{x} + 3\ln(x+1) - 3\ln 2$$

or, solving for y ,

$$y = -1 + \sqrt{\ln(x^2) + \frac{4}{x} + \ln[(x+1)^6] - 6\ln 2}.$$

22. Writing $2ydy = -x^2dx$ and integrating, we find

$$y^2 = -\frac{x^3}{3} + C.$$

With $y(0) = 2$,

$$(2)^2 = -\frac{(0)^3}{3} + C \quad \Rightarrow \quad C = 4,$$

and so

$$y^2 = -\frac{x^3}{3} + 4 \quad \Rightarrow \quad y = \sqrt{-\frac{x^3}{3} + 4}.$$

We note that, taking the square root, we chose the positive sign because $y(0) > 0$.

24. For a general solution, we separate variables and integrate.

$$\int e^{2y} dy = \int 8x^3 dx \quad \Rightarrow \quad \frac{e^{2y}}{2} = 2x^4 + C_1 \quad \Rightarrow \quad e^{2y} = 4x^4 + C.$$

We substitute now the initial condition, $y(1) = 0$, and obtain

$$1 = 4 + C \quad \Rightarrow \quad C = -3.$$

Hence, the answer is given by

$$e^{2y} = 4x^4 - 3 \quad \Rightarrow \quad y = \frac{1}{2} \ln(4x^4 - 3).$$

26. We separate variables and obtain

$$\int \frac{dy}{\sqrt{y}} = - \int \frac{dx}{1+x} \quad \Rightarrow \quad 2\sqrt{y} = -\ln|1+x| + C = -\ln(1+x) + C,$$

because at initial point, $x = 0$, $1+x > 0$. Using the fact that $y(0) = 1$, we find C .

$$2 = 0 + C \quad \Rightarrow \quad C = 2,$$

and so $y = [2 - \ln(1+x)]^2/4$ is the answer.

28. We have

$$\begin{aligned} \frac{dy}{dt} &= 2y(1-t) & \Rightarrow & \quad \frac{dy}{y} = 2(1-t)dt & \Rightarrow & \quad \ln|y| = -(t-1)^2 + C \\ \Rightarrow & \quad y = \pm e^C e^{-(t-1)^2} = C_1 e^{-(t-1)^2}, \end{aligned}$$

where $C_1 \neq 0$ is any constant. Separating variables, we lost the solution $y \equiv 0$, which can be included into the above formula by letting $C_1 = 0$. So, a general solution to the given equation is

$$y = C_1 e^{-(t-1)^2}, \quad C_1 \text{ is an arbitrary constant.}$$

Substituting $t = 0$ and $y = 3$, we find

$$3 = C_1 e^{-1} \quad \Rightarrow \quad C_1 = 3e \quad \Rightarrow \quad y = 3e^{1-(t-1)^2} = 3e^{2t-t^2}.$$

The graph of this function is given in Fig. **2-A** at the end of the chapter.

Since $y(t) > 0$ for any t , from the given equation we have $y'(t) > 0$ for $t < 1$ and $y'(t) < 0$ for $t > 1$. Thus $t = 1$ is the point of absolute maximum with $y_{\max} = y(1) = 3e$.

- 30. (a)** Dividing by $(y+1)^{2/3}$, multiplying by dx , and integrating, we obtain

$$\begin{aligned} \int \frac{dy}{(y+1)^{2/3}} &= \int (x-3)dx \quad \Rightarrow \quad 3(y+1)^{1/3} = \frac{x^2}{2} - 3x + C \\ \Rightarrow \quad y &= -1 + \left(\frac{x^2}{6} - x + C_1 \right)^3. \end{aligned}$$

- (b)** Substituting $y \equiv -1$ into the original equation yields

$$\frac{d(-1)}{dx} = (x-3)(-1+1)^{2/3} = 0,$$

and so the equation is satisfied.

- (c)** For $y \equiv -1$ for the solution in part (a), we must have

$$\left(\frac{x^2}{6} - x + C_1 \right)^3 \equiv 0 \quad \Leftrightarrow \quad \frac{x^2}{6} - x + C_1 \equiv 0,$$

which is impossible since a quadratic polynomial has at most two zeros.

- 32. (a)** The direction field of the given differential equation is shown in Fig. **2-B** at the end of the chapter. Using this picture we predict that $\lim_{x \rightarrow \infty} \phi(x) = 1$.
- (b)** In notation of Section 1.4, we have $x_0 = 0$, $y_0 = 1.5$, $f(x, y) = y^2 - 3y + 2$, and $h = 0.1$. With this step size, we need $(1-0)/0.1 = 10$ steps to approximate $\phi(1)$. The results of computation are given in Table **2-A** at the end of the chapter. From this table we conclude that $\phi(1) \approx 1.26660$.

(c) Separating variables and integrating, we obtain

$$\frac{dy}{y^2 - 3y + 2} = dx \quad \Rightarrow \quad \int \frac{dy}{y^2 - 3y + 2} = \int dx \quad \Rightarrow \quad \ln \left| \frac{y-2}{y-1} \right| = x + C,$$

where we have used a partial fractions decomposition

$$\frac{1}{y^2 - 3y + 2} = \frac{1}{y-2} - \frac{1}{y-1}$$

to evaluate the integral. The initial condition, $y(0) = 1.5$, implies that $C = 0$, and so

$$\ln \left| \frac{y-2}{y-1} \right| = x \quad \Rightarrow \quad \left| \frac{y-2}{y-1} \right| = e^x \quad \Rightarrow \quad \frac{y-2}{y-1} = -e^x.$$

(We have chosen the negative sign because of the initial condition.) Solving for y yields

$$y = \phi(x) = \frac{e^x + 2}{e^x + 1}$$

The graph of this solution is shown in Fig. **2-B** at the end of the chapter.

(d) We find

$$\phi(1) = \frac{e + 2}{e + 1} \approx 1.26894.$$

Thus, the approximate value $\phi(1) \approx 1.26660$ found in part (b) differs from the actual value by less than 0.003.

(e) We find the limit of $\phi(x)$ at infinity writing

$$\lim_{x \rightarrow \infty} \frac{e^x + 2}{e^x + 1} = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{e^x + 1} \right) = 1,$$

which confirms our guess in part (a).

34. (a) Separating variables and integrating, we get

$$\begin{aligned} \frac{dT}{T-M} = -kdt &\Rightarrow \int \frac{dT}{T-M} = -\int kdt \Rightarrow \ln |T-M| = -kt + C_1 \\ \Rightarrow |T-M| = e^{C_1} e^{-kt} &\Rightarrow T-M = \pm e^{C_1} e^{-kt} = C e^{-kt}, \end{aligned}$$

where C is any nonzero constant. We can include the lost solution $T \equiv M$ into this formula by letting $C = 0$. Thus, a general solution to the equation is

$$T = M + C e^{-kt}.$$

- (b) Given that $M = 70^\circ$, $T(0) = 100^\circ$, $T(6) = 80^\circ$, we form a system to determine C and k .

$$\begin{cases} 100 = 70 + C \\ 80 = 70 + Ce^{-6k} \end{cases} \Rightarrow \begin{cases} C = 30 \\ k = -(1/6) \ln[(80 - 70)/30] = (1/6) \ln 3. \end{cases}$$

Therefore,

$$T = 70 + 30e^{-(t \ln 3)/6} = 70 + (30)3^{-t/6},$$

and after 20 min the reading is

$$T(20) = 70 + (30)3^{-20/6} \approx 70.77^\circ.$$

36. A general solution to the cooling equation found in Problem 34, that is, $T = M + Ce^{-kt}$. Since $T(0) = 100^\circ$, $T(5) = 80^\circ$, and $T(10) = 65^\circ$, we determine M , C , and k from the system

$$\begin{cases} M + C = 100 \\ M + Ce^{-5k} = 80 \\ M + Ce^{-10k} = 65 \end{cases} \Rightarrow \begin{cases} C(1 - e^{-5k}) = 20 \\ Ce^{-5k}(1 - e^{-5k}) = 15 \end{cases} \Rightarrow e^{-5k} = 3/4.$$

To find M , we can now use the first two equations in the above system.

$$\begin{cases} M + C = 100 \\ M + (3/4)C = 80 \end{cases} \Rightarrow M = 20.$$

38. With $m = 100$, $g = 9.8$, and $k = 5$, the equation becomes

$$100 \frac{dv}{dt} = 100(9.8) - 5v \Rightarrow 20 \frac{dv}{dt} = 196 - v.$$

Separating variables and integrating yields

$$\int \frac{dv}{v - 196} = -\frac{1}{20} \int dt \Rightarrow \ln |v - 196| = -\frac{t}{20} + C_1 \Rightarrow v = 196 + Ce^{-t/20},$$

where C is an arbitrary nonzero constant. With $C = 0$, this formula also gives the (lost) constant solution $v(t) \equiv 196$. From the initial condition, $v(0) = 10$, we find C .

$$196 + C = 10 \Rightarrow C = -186 \Rightarrow v(t) = 196 - 186e^{-t/20}.$$

The terminal velocity of the object can be found by letting $t \rightarrow \infty$.

$$v_\infty = \lim_{t \rightarrow \infty} (196 - 186e^{-t/20}) = 196 \text{ (m/sec)}.$$

40. (a) Substituting $\rho = Mp/RT$ into $dp/dz = -\rho g$ yields

$$\frac{dp}{dz} = -\left(\frac{Mp}{RT}\right)g = -\frac{Mg}{RT}p.$$

Separating variables and integrating, we find

$$\frac{dp}{p} = -\frac{Mg}{RT}dz \quad (2.2)$$

$$\Rightarrow \ln p = -\frac{Mg}{RT}z + C_1 \quad \Rightarrow \quad p(z) = Ce^{-(Mg/RT)z}. \quad (2.3)$$

For $z = z_0$,

$$p(z_0) = Ce^{-(Mg/RT)z_0} \quad \Rightarrow \quad C = p(z_0)e^{(Mg/RT)z_0}.$$

Thus,

$$p(z) = p(z_0)e^{(Mg/RT)(z_0-z)} = p(z_0)e^{-Mg(z-z_0)/(RT)}.$$

(b) If $T = T(z)$ varies, then integrating (2.2) from z_0 to z we obtain

$$\begin{aligned} \ln \frac{p(z)}{p(z_0)} &= -\frac{Mg}{R} \int_{z_0}^z \frac{d\zeta}{T(\zeta)} \\ \Rightarrow \quad p(z) &= p(z_0) \exp \left[-\frac{Mg}{R} \int_{z_0}^z \frac{d\zeta}{T(\zeta)} \right]. \end{aligned}$$

(c) With the given formula for $T(z)$, the integral in part (b) gives

$$\int_{z_0}^z \frac{d\zeta}{288 - 0.0065(\zeta - z_0)} = -\frac{1}{0.0065} \ln \frac{288 - 0.0065(z - z_0)}{288}.$$

Therefore,

$$p(z) = p(z_0) \left[\frac{288 - 0.0065(z - z_0)}{288} \right]^{Mg/(0.0065R)}.$$

Substituting the given data into this equation and computing, we get the height

$$z - z_0 \approx 168 \text{ (m)}.$$

EXERCISES 2.3: Linear Equations

2. Neither.

4. Linear.

6. Linear.

8. Writing the equation in standard form,

$$\frac{dy}{dx} - \frac{y}{x} = 2x + 1,$$

we see that

$$P(x) = -\frac{1}{x} \quad \Rightarrow \quad \mu(x) = \exp \left[\int \left(-\frac{1}{x} \right) dx \right] = \exp(-\ln x) = \frac{1}{x}.$$

Multiplying the given equation by $\mu(x)$, we get

$$\frac{d}{dx} \left(\frac{y}{x} \right) = 2 + \frac{1}{x} \quad \Rightarrow \quad y = x \int \left(2 + \frac{1}{x} \right) dx = x(2x + \ln|x| + C).$$

10. From the standard form of the given equation,

$$\frac{dy}{dx} + \frac{2}{x}y = x^{-4},$$

we find that

$$\begin{aligned} \mu(x) &= \exp \left[\int (2/x) dx \right] = \exp(2 \ln|x|) = x^2 \\ \Rightarrow \quad \frac{d}{dx} (x^2 y) &= x^{-2} \quad \Rightarrow \quad y = x^{-2} \int x^{-2} dx = x^{-2} (-x^{-1} + C) = \frac{Cx - 1}{x^3}. \end{aligned}$$

12. Here, $P(x) = 4$, $Q(x) = x^2 e^{-4x}$. So, $\mu(x) = e^{4x}$ and

$$\frac{d}{dx} (e^{4x} y) = x^2 \quad \Rightarrow \quad y = e^{-4x} \int x^2 dx = e^{-4x} \left(\frac{x^3}{3} + C \right).$$

14. In standard form, we have

$$\frac{dy}{dx} + \frac{3}{x}y = \frac{\sin x}{x^2} - 3x.$$

Therefore, $P(x) = 3/x$,

$$\mu(x) = \exp \left(\int \frac{3 dx}{x} \right) = x^3$$

and

$$y = \frac{1}{x^3} \int (x \sin x - 3x^4) dx = \frac{1}{x^3} \left(\sin x - x \cos x - \frac{3}{5} x^5 + C \right).$$

(Integration by parts was used to integrate the first term.)

16. Here, $|x| < 1$ and

$$P(x) = -\frac{x^2}{1-x^2} = 1 - \frac{1}{1-x^2}, \quad Q(x) = \frac{1+x}{\sqrt{1-x^2}}.$$

Therefore,

$$\mu(x) = \exp \left[\int \left(1 - \frac{1}{1-x^2} \right) dx \right] = e^x \sqrt{\frac{1-x}{1+x}}$$

so that $\mu(x)Q(x) = e^x$ and

$$y = e^{-x} \sqrt{\frac{1+x}{1-x}} (e^x + C) = \sqrt{\frac{1+x}{1-x}} (1 + Ce^{-x}).$$

18. Since $\mu(x) = \exp(\int 4dx) = e^{4x}$, we have

$$\begin{aligned} \frac{d}{dx} (e^{4x}y) &= e^{4x}e^{-x} = e^{3x} \\ \Rightarrow y &= e^{-4x} \int e^{3x} dx = \frac{e^{-x}}{3} + Ce^{-4x}. \end{aligned}$$

Substituting the initial condition, $y = 4/3$ at $x = 0$, yields

$$\frac{4}{3} = \frac{1}{3} + C \quad \Rightarrow \quad C = 1,$$

and so $y = e^{-x}/3 + e^{-4x}$ is the solution to the given initial value problem.

20. We have

$$\begin{aligned} \mu(x) &= \exp \left(\int \frac{3dx}{x} \right) = \exp(3 \ln x) = x^3 \\ \Rightarrow x^3 y &= \int x^3 (3x - 2) dx = \frac{3x^5}{5} - \frac{x^4}{2} + C \\ \Rightarrow y &= \frac{3x^2}{5} - \frac{x}{2} + Cx^{-3}. \end{aligned}$$

With $y(1) = 1$,

$$1 = y(1) = \frac{3}{5} - \frac{1}{2} + C \quad \Rightarrow \quad C = \frac{9}{10} \quad \Rightarrow \quad y = \frac{3x^2}{5} - \frac{x}{2} + \frac{9}{10x^3}.$$

22. From the standard form of this equation,

$$\frac{dy}{dx} + y \cot x = x,$$

we find

$$\mu(x) = \exp \left(\int \cot x dx \right) = \exp(\ln \sin x) = \sin x.$$

(Alternatively, one can notice that the left-hand side of the original equation is the derivative of the product $y \sin x$.) So, using integration by parts, we obtain

$$\begin{aligned} y \sin x &= \int x \sin x \, dx = -x \cos x + \sin x + C \\ \Rightarrow y &= -x \cot x + 1 + C \csc x. \end{aligned}$$

We find C using the initial condition $y(\pi/2) = 2$:

$$2 = -\frac{\pi}{2} \cot \frac{\pi}{2} + 1 + C \csc \left(\frac{\pi}{2} \right) = 1 + C \quad \Rightarrow \quad C = 1,$$

and the solution is given by

$$y = -x \cot x + 1 + \csc x.$$

24. (a) The equation (12) of the text becomes

$$\begin{aligned} \frac{dy}{dt} + 20y &= 50e^{-10t} \quad \Rightarrow \quad \mu(t) = e^{20t} \\ \Rightarrow y &= e^{-20t} \int e^{20t} (50e^{-10t}) \, dt = e^{-20t} \int 50e^{10t} \, dt = 5e^{-10t} + Ce^{-20t}. \end{aligned}$$

Since $y(0) = 40$, we have

$$40 = 5 + C \quad \Rightarrow \quad C = 35 \quad \Rightarrow \quad y = 5e^{-10t} + 35e^{-20t}.$$

The term $5e^{-10t}$ will eventually dominate.

(b) This time, the equation (12) has the form

$$\begin{aligned} \frac{dy}{dt} + 10y &= 50e^{-10t} \quad \Rightarrow \quad \mu(t) = e^{10t} \\ \Rightarrow y &= e^{-10t} \int e^{10t} (50e^{-10t}) \, dt = e^{-10t}(50t + C). \end{aligned}$$

Substituting the initial condition yields

$$40 = y(0) = C \quad \Rightarrow \quad y = e^{-10t}(50t + 40).$$

26. Here

$$\begin{aligned} P(x) &= \frac{\sin x \cos x}{1 + \sin^2 x} \\ \Rightarrow \mu(x) &= \exp \left(\int \frac{\sin x \cos x \, dx}{1 + \sin^2 x} \right) = \exp \left[\frac{1}{2} \ln (1 + \sin^2 x) \right] = \sqrt{1 + \sin^2 x}. \end{aligned}$$

Thus,

$$y\sqrt{1+\sin^2 x} = \int_0^x \sqrt{1+\sin^2 t} dt \Rightarrow y = (1+\sin^2 x)^{-1/2} \int_0^x (1+\sin^2 t)^{1/2} dt$$

and

$$y(1) = (1+\sin^2 1)^{-1/2} \int_0^1 (1+\sin^2 t)^{1/2} dt.$$

We now use the Simpson's Rule to find that $y(1) \approx 0.860$.

28. (a) Substituting $y = e^{-x}$ into the equation (16) yields

$$\frac{d(e^{-x})}{dx} + e^{-x} = -e^{-x} + e^{-x} = 0.$$

So, $y = e^{-x}$ is a solution to (16).

The function $y = x^{-1}$ is a solution to (17) because

$$\frac{d(x^{-1})}{dx} + (x^{-1})^2 = (-1)x^{-2} + x^{-2} = 0.$$

- (b) For any constant C ,

$$\frac{d(Ce^{-x})}{dx} + Ce^{-x} = -Ce^{-x} + Ce^{-x} = 0.$$

Thus $y = Ce^{-x}$ is a solution to (16).

Substituting $y = Cx^{-1}$ into (17), we obtain

$$\frac{d(Cx^{-1})}{dx} + (Cx^{-1})^2 = (-C)x^{-2} + C^2x^{-2} = C(C-1)x^{-2},$$

and so we must have $C(C-1) = 0$ in order that $y = Cx^{-1}$ is a solution to (17).

Thus, either $C = 0$ or $C = 1$.

- (c) For the function $y = C\hat{y}$, one has

$$\frac{d(C\hat{y})}{dx} + P(x)(C\hat{y}) = C\frac{d\hat{y}}{dx} + C(P(x)\hat{y}) = C\left(\frac{d\hat{y}}{dx} + P(x)\hat{y}\right) = 0$$

if $\hat{y}$ is a solution to $y' + P(x)y = 0$.

30. (a) Multiplying both sides of (18) by y^2 , we get

$$y^2 \frac{dy}{dx} + 2y^3 = x.$$

If $v = y^3$, then $v' = 3y^2y'$. Thus, $y^2y' = v'/3$, and we have

$$\frac{1}{3} \frac{dv}{dx} + 2v = x,$$

which is equivalent to (19).

(b) The equation (19) is linear with $P(x) = 6$ and $Q(x) = 3x$. So,

$$\begin{aligned} \mu(x) &= \exp\left(\int 6dx\right) = e^{6x} \\ \Rightarrow \quad v(x) &= e^{-6x} \int (3xe^{6x})dx = \frac{e^{-6x}}{2} \left(xe^{6x} - \int e^{6x}dx\right) \\ &= \frac{e^{-6x}}{2} \left(xe^{6x} - \frac{e^{6x}}{6} + C_1\right) = \frac{x}{2} - \frac{1}{12} + Ce^{-6x}, \end{aligned}$$

where $C = C_1/2$ is an arbitrary constant. The back substitution yields

$$y = \sqrt[3]{\frac{x}{2} - \frac{1}{12} + Ce^{-6x}}.$$

32. In the given equation, $P(x) = 2$, which implies that $\mu(x) = e^{2x}$. Following guidelines, first we solve the equation on $[0, 3]$. On this interval, $Q(x) \equiv 2$. Therefore,

$$y_1(x) = e^{-2x} \int (2)e^{2x}dx = 1 + C_1e^{-2x}.$$

Since $y_1(0) = 0$, we get

$$1 + C_1e^0 = 0 \quad \Rightarrow \quad C_1 = -1 \quad \Rightarrow \quad y_1(x) = 1 - e^{-2x}.$$

For $x > 3$, $Q(x) = -2$ and so

$$y_2(x) = e^{-2x} \int (-2)e^{2x}dx = -1 + C_2e^{-2x}.$$

We now choose C_2 so that

$$y_2(3) = y_1(3) = 1 - e^{-6} \quad \Rightarrow \quad -1 + C_2e^{-6} = 1 - e^{-6} \quad \Rightarrow \quad C_2 = 2e^6 - 1.$$

Therefore, $y_2(x) = -1 + (2e^6 - 1)e^{-2x}$, and the continuous solution to the given initial value problem on $[0, \infty)$ is

$$y(x) = \begin{cases} 1 - e^{-2x}, & 0 \leq x \leq 3, \\ -1 + (2e^6 - 1)e^{-2x}, & x > 3. \end{cases}$$

The graph of this function is shown in Fig. **2-D** at the end of the chapter.

- 34. (a)** Since $P(x)$ is continuous on (a, b) , its antiderivatives given by $\int P(x)dx$ are continuously differentiable, and therefore continuous, functions on (a, b) . Since the function e^x is continuous on $(-\infty, \infty)$, composite functions $\mu(x) = e^{\int P(x)dx}$ are continuous on (a, b) . The range of the exponential function is $(0, \infty)$. This implies that $\mu(x)$ is positive with any choice of the integration constant. Using the chain rule, we conclude that

$$\frac{d\mu(x)}{dx} = e^{\int P(x)dx} \frac{d}{dx} \left(\int P(x)dx \right) = \mu(x)P(x)$$

for any x on (a, b) .

- (b)** Differentiating (8), we apply the product rule and obtain

$$\frac{dy}{dx} = -\mu^{-2}\mu' \left(\int \mu Q dx + C \right) + \mu^{-1}\mu Q = -\mu^{-1}P \left(\int \mu Q dx + C \right) + Q,$$

and so

$$\frac{dy}{dx} + Py = \left[-\mu^{-1}P \left(\int \mu Q dx + C \right) + Q \right] + P \left[\mu^{-1} \left(\int \mu Q dx + C \right) \right] = Q.$$

- (c)** Suggested choice of the antiderivative and the constant C yields

$$y(x_0) = \mu(x)^{-1} \left(\int_{x_0}^x \mu Q dx + y_0 \mu(x_0) \right) \Big|_{x=x_0} = \mu(x_0)^{-1} y_0 \mu(x_0) = y_0.$$

- (d)** We assume that $y(x)$ is a solution to the initial value problem (15). Since $\mu(x)$ is a continuous positive function on (a, b) , the equation (5) is equivalent to (4). Since, from the part (a), the left-hand side of (5) is the derivative of the product $\mu(x)y(x)$, this function must be an antiderivative of the right-hand side, which is $\mu(x)Q(x)$. Thus, we come up with (8), where the integral means one of the antiderivatives, for example, the one suggested in the part (c) (which has zero value at x_0). Substituting $x = x_0$ into (8), we conclude that

$$y_0 = y(x_0) = \mu(x_0)^{-1} \left(\int \mu Q dx + C \right) \Big|_{x=x_0} = C \mu(x_0)^{-1},$$

and so $C = y_0 \mu(x_0)$ is uniquely defined.

- 36. (a)** If $\mu(x) = \exp \left(\int P dx \right)$ and $y_h(x) = \mu(x)^{-1}$, then

$$\frac{dy_h}{dx} = (-1)\mu(x)^{-2} \frac{d\mu(x)}{dx} = -\mu(x)^{-2} \mu(x)P(x) = -\mu(x)^{-1}P(x)$$

and so

$$\frac{dy_h}{dx} + P(x)y_h = -\mu(x)^{-1}P(x) + P(x)\mu(x)^{-1} = 0,$$

i.e., y_h is a solution to the equation $y' + Py = 0$. Now, the formula (8) yields

$$y = \mu(x)^{-1} \left(\int \mu(x)Q(x)dx + C \right) = y_h(x)v(x) + Cy_h(x) = y_p(x) + Cy_h(x),$$

where $v(x) = \int \mu(x)Q(x)dx$.

(b) Separating variables in (22) and integrating, we obtain

$$\frac{dy}{y} = -\frac{3dx}{x} \quad \Rightarrow \quad \int \frac{dy}{y} = -\int \frac{3dx}{x} \quad \Rightarrow \quad \ln|y| = -3\ln x + C.$$

Since we need just one solution y_h , we take $C = 0$

$$\ln|y| = -3\ln x \quad \Rightarrow \quad y = \pm x^{-3},$$

and we choose, say, $y_h = x^{-3}$.

(c) Substituting $y_p = v(x)y_h(x) = v(x)x^{-3}$ into (21), we get

$$\frac{dv}{dx}y_h + v\frac{dy_h}{dx} + \frac{3}{x}vy_h = \frac{dv}{dx}y_h + v\left(\frac{dy_h}{dx} + \frac{3}{x}y_h\right) = \frac{dv}{dx}y_h = x^2.$$

Therefore, $dv/dx = x^2/y_h = x^5$.

(d) Integrating yields

$$v(x) = \int x^5 dx = \frac{x^6}{6}.$$

(We have chosen zero integration constant.)

(e) The function

$$y = Cy_h + vy_h = Cx^{-3} + \frac{x^3}{6}$$

is a general solution to (21) because

$$\begin{aligned} \frac{dy}{dx} + \frac{3}{x}y &= \frac{d}{dx} \left(Cx^{-3} + \frac{x^3}{6} \right) + \frac{3}{x} \left(Cx^{-3} + \frac{x^3}{6} \right) \\ &= \left(-3Cx^{-4} + \frac{x^2}{2} \right) + \left(3Cx^{-4} + \frac{x^2}{2} \right) = x^2. \end{aligned}$$

38. Dividing both sides of (6) by μ and multiplying by dx yields

$$\begin{aligned} \frac{d\mu}{\mu} &= Pdx \quad \Rightarrow \quad \int \frac{d\mu}{\mu} = \int Pdx \\ \Rightarrow \quad \ln|\mu| &= \int Pdx \quad \Rightarrow \quad \mu = \pm \exp \left(\int Pdx \right). \end{aligned}$$

Choosing the positive sign, we obtain (7).

40. (a) Figure 2-E at the end of this chapter depicts the direction field for $\alpha = \frac{1}{4}, \beta = \frac{1}{2}$. If $u = 1, u' = -\beta < 0$; if $u = 0, u' = \alpha > 0$. Thus $u(t)$ is confined to $[0, 1]$ if it ever enters this interval.

(b) The general solution $u(t) = \frac{\alpha}{\alpha+\beta} + Ce^{-(\alpha+\beta)t}$ approaches $\frac{\alpha}{\alpha+\beta}$ as $t \rightarrow \infty$.

EXERCISES 2.4: Exact Equations

2. This equation is not separable because the coefficient $x^{10/3} - 2y$ cannot be written as a product $p(x)q(y)$. Writing the equation in the form

$$\frac{dy}{dx} - 2x^{-1}y = -x^{7/3},$$

we see that the equation is linear. Since $M(x, y) = x^{10/3} - 2y$, $N(x, y) = x$,

$$\frac{\partial M}{\partial y} = -2 \neq \frac{\partial N}{\partial x} = 1,$$

and so the equation is not exact.

4. Here $M(x, y) = ye^{xy} + 2x$, $N(x, y) = xe^{xy} - 2y$. Thus

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(ye^{xy} + 2x) = e^{xy} + y \frac{\partial}{\partial y}(e^{xy}) = e^{xy} + ye^{xy}x = e^{xy}(1 + yx), \\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(xe^{xy} - 2y) = e^{xy} + x \frac{\partial}{\partial x}(e^{xy}) = e^{xy} + xe^{xy}y = e^{xy}(1 + xy),\end{aligned}$$

Since $\partial M/\partial y = \partial N/\partial x$, the equation is exact.

Writing the equation in the form

$$\frac{dy}{dx} = -\frac{ye^{xy} + 2x}{xe^{xy} - 2y},$$

we conclude that it is not separable, because the right-hand side cannot be represented as a product of two functions of single variables x and y . Also, the right-hand side is not linear with respect to y , which implies that the equation is not linear with y as the dependent variable. Similarly, choosing x as the dependent variable (taking the reciprocals of both sides), we conclude that the equation is not linear in x either.

6. The differential equation is not separable because $(2xy + \cos y)$ cannot be factored. This equation can be put in standard form by defining x as the dependent variable and y as the independent variable. This gives

$$\frac{dx}{dy} + \frac{2}{y}x = \frac{-\cos y}{y^2},$$

and so we see that it is linear.

If we set $M(x, y) = y^2$ and $N(x, y) = 2xy + \cos y$, we see that the differential equation is also exact, because $M_y(x, y) = 2y = N_x(x, y)$.

8. In this problem, the variables are r and θ , $M(r, \theta) = \theta$, and $N(r, \theta) = 3r - \theta - 1$. Since

$$\frac{\partial M}{\partial \theta} = 1 \neq 3 = \frac{\partial N}{\partial r},$$

the equation is not exact. With r as the dependent variable, the equation takes the form

$$\frac{dr}{d\theta} = -\frac{3r - \theta - 1}{\theta} = -\frac{3}{\theta}r + \frac{\theta + 1}{\theta},$$

and so it is linear. Since the right-hand side in the above equation cannot be factored as $p(\theta)q(r)$, the equation is not separable.

10. In this problem, $M(x, y) = 2x + y$, $N(x, y) = x - 2y$. Thus, $M_y = N_x = 1$, and the equation is exact. We find

$$\begin{aligned} F(x, y) &= \int (2x + y)dx = x^2 + xy + g(y), \\ \frac{\partial F}{\partial y} &= x + g'(y) = N(x, y) = x - 2y \\ \Rightarrow \quad g'(y) &= -2y \quad \Rightarrow \quad g(y) = \int (-2y)dy = -y^2 \\ \Rightarrow \quad F(x, y) &= x^2 + xy - y^2, \end{aligned}$$

and so $x^2 + xy - y^2 = C$ is a general solution.

12. Taking partial derivatives of $M(x, y) = \cos x \cos y + 2x$ and $N(x, y) = -(\sin x \sin y + 2y)$, we obtain

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (\cos x \cos y + 2x) = -\cos x \sin y, \\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} [-(\sin x \sin y + 2y)] = -\cos x \sin y, \end{aligned}$$

and so the equation is exact.

Integrating $M(x, y)$ with respect to x yields

$$F(x, y) = \int M(x, y)dx = \int (\cos x \cos y + 2x) dx$$

$$= \cos y \int \cos x \, dx + \int 2x \, dx = \sin x \cos y + x^2 + g(y).$$

To find $g(y)$, we compute the partial derivative of $F(x, y)$ with respect to y and compare the result with $N(x, y)$.

$$\begin{aligned} \frac{\partial F}{\partial y} &= \frac{\partial}{\partial y} [\sin x \cos y + x^2 + g(y)] = -\sin x \sin y + g'(y) = -(\sin x \sin y + 2y) \\ \Rightarrow \quad g'(y) &= -2y \quad \Rightarrow \quad g(y) = \int (-2y) dy = -y^2. \end{aligned}$$

(We have taken zero integration constant.) Therefore,

$$F(x, y) = \sin x \cos y + x^2 - y^2 = C$$

is a general solution to the given equation.

14. In this equation, the variables are y and t , $M(y, t) = t/y$, $N(y, t) = 1 + \ln y$. Since

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t} \left(\frac{t}{y} \right) = \frac{1}{y} \quad \text{and} \quad \frac{\partial N}{\partial y} = \frac{\partial}{\partial y} (1 + \ln y) = \frac{1}{y},$$

the equation is exact.

Integrating $M(y, t)$ with respect to y , we get

$$F(y, t) = \int \frac{t}{y} dy = t \ln |y| + g(t) = t \ln y + g(t).$$

(From $N(y, t) = 1 + \ln y$ we conclude that $y > 0$.) Therefore,

$$\begin{aligned} \frac{\partial F}{\partial t} &= \frac{\partial}{\partial t} [t \ln y + g(t)] = \ln y + g'(t) = 1 + \ln y \\ \Rightarrow \quad g'(t) &= 1 \quad \Rightarrow \quad g(t) = t \\ \Rightarrow \quad F(y, t) &= t \ln y + t, \end{aligned}$$

and a general solution is given by $t \ln y + t = C$ (or, explicitly, $t = C/(\ln y + 1)$).

16. Computing

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (ye^{xy} - y^{-1}) = e^{xy} + xy e^{xy} + y^{-2}, \\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} (xe^{xy} + xy^{-2}) = e^{xy} + xy e^{xy} + y^{-2}, \end{aligned}$$

we see that the equation is exact. Therefore,

$$F(x, y) = \int (ye^{xy} - y^{-1}) dx = e^{xy} - xy^{-1} + g(y).$$

So,

$$\frac{\partial F}{\partial y} = xe^{xy} + xy^{-2} + g'(y) = N(x, y) \quad \Rightarrow \quad g'(y) = 0.$$

Thus, $g(y) = 0$, and the answer is $e^{xy} - xy^{-1} = C$.

18. Since

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2y^2 + \sin(x + y),$$

the equation is exact. We find

$$\begin{aligned} F(x, y) &= \int [2x + y^2 - \cos(x + y)] dx = x^2 + xy^2 - \sin(x + y) + g(y), \\ \frac{\partial F}{\partial y} &= 2xy - \cos(x + y) + g'(y) = 2xy - \cos(x + y) - e^y \\ \Rightarrow \quad g'(y) &= -e^y \quad \Rightarrow \quad g(y) = -e^y. \end{aligned}$$

Therefore,

$$F(x, y) = x^2 + xy^2 - \sin(x + y) - e^y = C$$

gives a general solution.

20. We find

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} [y \cos(xy)] = \cos(xy) - xy \sin(xy), \\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} [x \cos(xy)] = \cos(xy) - xy \sin(xy). \end{aligned}$$

Therefore, the equation is exact and

$$\begin{aligned} F(x, y) &= \int (x \cos(xy) - y^{-1/3}) dy = \sin(xy) - \frac{3}{2} y^{2/3} + h(x) \\ \frac{\partial F}{\partial x} &= y \cos(xy) + h'(x) = \frac{2}{\sqrt{1-x^2}} + y \cos(xy) \\ \Rightarrow \quad h'(x) &= \frac{2}{\sqrt{1-x^2}} \quad \Rightarrow \quad h(x) = 2 \arcsin x, \end{aligned}$$

and a general solution is given by

$$\sin(xy) - \frac{3}{2} y^{2/3} + 2 \arcsin x = C.$$

22. In Problem 16, we found that a general solution to this equation is

$$e^{xy} - xy^{-1} = C.$$

Substituting the initial condition, $y(1) = 1$, yields $e - 1 = C$. So, the answer is

$$e^{xy} - xy^{-1} = e - 1.$$

24. First, we check the given equation for exactness.

$$\frac{dM}{dx} = e^t = \frac{\partial N}{\partial t}.$$

So, it is exact. We find

$$\begin{aligned} F(t, x) &= \int (e^t - 1) dx = x(e^t - 1) + g(t), \\ \frac{\partial F}{\partial t} &= xe^t + g'(t) = xe^t + 1 \quad \Rightarrow \quad g(t) = \int dt = t \\ \Rightarrow \quad x(e^t - 1) + t &= C \end{aligned}$$

is a general solution. With $x(1) = 1$, we get

$$(1)(e - 1) + 1 = C \quad \Rightarrow \quad C = e,$$

and the solution is given by

$$x = \frac{e - t}{e^t - 1}.$$

26. Taking partial derivatives M_y and N_x , we find that the equation is exact. So,

$$\begin{aligned} F(x, y) &= \int (\tan y - 2) dx = x(\tan y - 2) + g(y), \\ \frac{\partial F}{\partial y} &= x \sec^2 y + g'(y) = x \sec^2 y + y^{-1} \\ \Rightarrow \quad g'(y) &= y^{-1} \quad \Rightarrow \quad g(y) = \ln |y|, \end{aligned}$$

and

$$x(\tan y - 2) + \ln |y| = C$$

is a general solution. Substituting $y(0) = 1$ yields $C = 0$. Therefore, the answer is

$$x(\tan y - 2) + \ln y = 0.$$

(We removed the absolute value sign in the logarithmic function because $y(0) > 0$.)

28. (a) Computing

$$\frac{\partial M}{\partial y} = \cos(xy) - xy \sin(xy),$$

which must be equal to $\partial N / \partial x$, we find that

$$\begin{aligned} N(x, y) &= \int [\cos(xy) - xy \sin(xy)] dx \\ &= \int [x \cos(xy)]'_x dx = x \cos(xy) + g(y). \end{aligned}$$

(b) Since

$$\frac{\partial M}{\partial y} = (1 + xy)e^{xy} - 4x^3 = \frac{\partial N}{\partial x},$$

we conclude that

$$N(x, y) = \int [(1 + xy)e^{xy} - 4x^3] dx = xe^{xy} - x^4 + g(y).$$

30. (a) Differentiating, we find that

$$\begin{aligned}\frac{\partial M}{\partial y} &= 5x^2 + 12x^3y + 8xy, \\ \frac{\partial N}{\partial x} &= 6x^2 + 12x^3y + 6xy.\end{aligned}$$

Since $M_y \neq N_x$, the equation is not exact.

(b) Multiplying given equation by $x^n y^m$ and taking partial derivatives of new coefficients yields

$$\begin{aligned}\frac{d}{dy} (5x^{n+2}y^{m+1} + 6x^{n+3}y^{m+2} + 4x^{n+1}y^{m+2}) \\ &= 5(m+1)x^{n+2}y^m + 6(m+2)x^{n+3}y^{m+1} + 4(m+2)x^{n+1}y^{m+1} \\ \frac{d}{dx} (2x^{n+3}y^m + 3x^{n+4}y^{m+1} + 3x^{n+2}y^{m+1}) \\ &= 2(n+3)x^{n+2}y^m + 3(n+4)x^{n+3}y^{m+1} + 3(n+2)x^{n+1}y^{m+1}.\end{aligned}$$

In order that these polynomials are equal, we must have equal coefficients at similar monomials. Thus, n and m must satisfy the system

$$\begin{cases} 5(m+1) = 2(n+3) \\ 6(m+2) = 3(n+4) \\ 4(m+2) = 3(n+2). \end{cases}$$

Solving, we obtain $n = 2$ and $m = 1$. Therefore, multiplying the given equation by x^2y yields an exact equation.

(c) We find

$$\begin{aligned}F(x, y) &= \int (5x^4y^2 + 6x^5y^3 + 4x^3y^3) dx \\ &= x^5y^2 + x^6y^3 + x^4y^3 + g(y).\end{aligned}$$

Therefore,

$$\frac{\partial F}{\partial y} = 2x^5y + 3x^6y^2 + 3x^4y^2 + g'(y)$$

$$= 2x^5y + 3x^6y^2 + 3x^4y^2 \Rightarrow g(y) = 0,$$

and a general solution to the given equation is

$$x^5y^2 + x^6y^3 + x^4y^3 = C.$$

- 32. (a)** The slope of the orthogonal curves, say $m_{\perp}$, must be $-1/m$, where m is the slope of the original curves. Therefore, we have

$$m_{\perp} = \frac{F_y(x, y)}{F_x(x, y)} \Rightarrow \frac{dy}{dx} = \frac{F_y(x, y)}{F_x(x, y)} \Rightarrow F_y(x, y) dx - F_x(x, y) dy = 0.$$

- (b)** Let $F(x, y) = x^2 + y^2$. Then we have $F_x(x, y) = 2x$ and $F_y(x, y) = 2y$. Substituting these expressions into the result of part (a) gives us

$$2y dx - 2x dy = 0 \Rightarrow y dx - x dy = 0.$$

To find the orthogonal trajectories, we must solve this differential equation. To this end, note that this equation is separable and thus

$$\begin{aligned} \int \frac{1}{x} dx &= \int \frac{1}{y} dy \Rightarrow \ln|x| = \ln|y| + C \\ \Rightarrow e^{\ln|x| - C} &= e^{\ln|y|} \Rightarrow y = kx, \text{ where } k = \pm e^{-C}. \end{aligned}$$

Therefore, the orthogonal trajectories are lines through the origin.

- (c)** Let $F(x, y) = xy$. Then we have $F_x(x, y) = y$ and $F_y(x, y) = x$. Substituting these expressions into the result of part (a) yields

$$x dx - y dy = 0.$$

To find the orthogonal trajectories, we must solve this differential equation. To this end, note that this equation is separable and thus

$$\int x dx = \int y dy \Rightarrow \frac{x^2}{2} = \frac{y^2}{2} + C \Rightarrow x^2 - y^2 = k,$$

where $k := 2C$. Therefore, the orthogonal trajectories are hyperbolas.

- 34.** To use the method described in Problem 32, we rewrite the equation $x^2 + y^2 = kx$ in the form $x + x^{-1}y^2 = k$. Thus, $F(x, y) = x + x^{-1}y^2$,

$$\frac{\partial F}{\partial x} = 1 - x^{-2}y^2, \quad \frac{\partial F}{\partial y} = 2x^{-1}y.$$

Substituting these derivatives in the equation given in Problem 32(b), we get the required. Multiplying the equation by $x^n y^m$, we obtain

$$2x^{n-1}y^{m+1}dx + (x^{n-2}y^{m+2} - x^n y^m) dy = 0.$$

Therefore,

$$\begin{aligned}\frac{\partial M}{\partial y} &= 2(m+1)x^{n-1}y^m, \\ \frac{\partial N}{\partial x} &= (n-2)x^{n-3}y^{m+2} - nx^{n-1}y^m.\end{aligned}$$

Thus, to have an exact equation, n and m must satisfy

$$\begin{cases} n-2=0 \\ 2(m+1)=-n. \end{cases}$$

Solving, we obtain $n=2$, $m=-2$. With this choice, the equation becomes

$$2xy^{-1}dx + (1 - x^2y^{-2}) dy = 0,$$

and so

$$\begin{aligned}G(x, y) &= \int M(x, y)dx = \int 2xy^{-1}dx = x^2y^{-1} + g(y), \\ \frac{\partial G}{\partial y} &= -x^2y^{-2} + g'(y) = N(x, y) = 1 - x^2y^{-2}.\end{aligned}$$

Therefore, $g(y) = y$, and the family of orthogonal trajectories is given by $x^2y^{-1} + y = C$. Writing this equation in the form $x^2 + y^2 - Cy = 0$, we see that, given C , the trajectory is the circle centered at $(0, C/2)$ and of radius $C/2$.

Several given curves and their orthogonal trajectories are shown in Fig. 2–E at the end of the chapter.

- 36.** The first equation in (4) follows from (9) and the Fundamental Theorem of Calculus.

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x} \left[\int_{x_0}^x M(t, y)dt + g(y) \right] = M(t, y)|_{t=x} = M(x, y).$$

For the second equation in (4),

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y} \left[\int_{x_0}^x M(t, y)dt + g(y) \right]$$

$$\begin{aligned}
&= \frac{\partial}{\partial y} \int_{x_0}^x M(t, y) dt + g'(y) \\
&= \frac{\partial}{\partial y} \int_{x_0}^x M(t, y) dt + \left[N(x, y) - \frac{\partial}{\partial y} \int_{x_0}^x M(t, y) dt \right] = N(x, y).
\end{aligned}$$

EXERCISES 2.5: Special Integrating Factors

2. Here, $M(x, y) = 2y^3 + 2y^2$ and $N(x, y) = 3y^2x + 2xy$. Computing

$$\frac{\partial M}{\partial y} = 6y^2 + 4y \quad \text{and} \quad \frac{\partial N}{\partial x} = 3y^2 + 2y,$$

we conclude that this equation is not exact. Note that these derivatives, as well as M itself, depend on y only. Then, clearly, so does the expression $(\partial N/\partial x - \partial M/\partial y)/M$, and the given equation has an integrating factor depending on y alone. Also, since

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{(6y^2 + 4y) - (3y^2 + 2y)}{3y^2x + 2xy} = \frac{3y^2 + 2y}{x(3y^2 + 2y)} = \frac{1}{x},$$

the equation has an integrating factor depending on x only.

Writing the equation in the form

$$\frac{dx}{dy} = -\frac{3y^2x + 2xy}{2y^3 + y^2} = -\frac{xy(3y + 2)}{2y^2(y + 1)} = -\frac{y(3y + 2)}{2y^2(y + 1)}x,$$

we conclude that it is separable and linear with x as the dependent variable.

4. This equation is not separable, because of the factor $(y^2 + 2xy)$. It is not linear in either variable because of the terms y^2 and x^2 . To see if it is exact, we compute $M_y(x, y)$ and $N_x(x, y)$, and find that

$$M_y(x, y) = 2y + 2x \neq -2x = N_x(x, y).$$

Therefore, the equation is not exact. To see if we can find an integrating factor of the form $\mu(x)$, we compute

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{2y + 4x}{-x^2},$$

which is not a function of x alone. To see if we can find an integrating factor of the form $\mu(y)$, we compute

$$\frac{\partial N/\partial x - \partial M/\partial y}{M} = \frac{-4x - 2y}{y^2 + 2xy} = \frac{-2(2x + y)}{y(y + 2x)} = \frac{-2}{y}.$$

Thus the equation has an integrating factor that is a function of y alone.

6. In this problem, $M(x, y) = 2y^2x - y$ and $N(x, y) = x$. Therefore,

$$\frac{\partial M}{\partial y} = 4yx - 1 \quad \text{and} \quad \frac{\partial N}{\partial x} = 1 \quad \Rightarrow \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2 - 4yx.$$

The equation is not exact, because $\partial M/\partial y \neq \partial N/\partial x$, but it has an integrating factor, which depends on y because

$$\frac{\partial N/\partial x - \partial M/\partial y}{M} = \frac{2 - 4yx}{2y^2x - y} = \frac{-2(2yx - 1)}{y(2yx - 1)} = \frac{-2}{y}.$$

Isolating dy/dx , we obtain

$$\frac{dy}{dx} = \frac{y - 2y^2x}{x} = \frac{y}{x} - 2y^2.$$

The right-hand side cannot be written as $p(x)q(y)$, so the equation is not separable. Also, it is not linear with y as the dependent variable (because of $2y^2$ term). Taking the reciprocals, we conclude that it is not linear with the dependent variable x .

8. The equation $(3x^2 + y)dx + (x^2y - x)dy = 0$ is not separable or linear. To see if it is exact, we compute

$$\frac{\partial M}{\partial y} = 1 \neq 2xy - 1 = \frac{\partial N}{\partial x}.$$

Thus, the equation is not exact. To see if we can find an integrating factor, we compute

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{2 - 2xy}{x^2y - x} = \frac{-2(xy - 1)}{x(xy - 1)} = \frac{-2}{x}.$$

Thus, an integrating factor $\mu(x)$ is given by

$$\mu(x) = \exp\left(\int \frac{-2}{x} dx\right) = \exp(-2 \ln |x|) = x^{-2}.$$

To solve the equation, we multiply it by the integrating factor x^{-2} and get

$$(3 + yx^{-2})dx + (y - x^{-1})dy = 0.$$

This new equation is now exact. Thus, we find $F(x, y)$ integrating $M(x, y) = 3 + yx^{-2}$ with respect to x .

$$\begin{aligned} F(x, y) &= \int (3 + yx^{-2}) dx = 3x - yx^{-1} + g(y) \\ \Rightarrow F_y(x, y) &= -x^{-1} + g'(y) = N(x, y) = y - x^{-1} \\ \Rightarrow g'(y) &= y \quad \Rightarrow \quad g(y) = \frac{y^2}{2}. \end{aligned}$$

Therefore,

$$F(x, y) = 3x - yx^{-1} + \frac{y^2}{2},$$

and an implicit solution to the given equation is

$$\frac{y^2}{2} - \frac{y}{x} + 3x = C.$$

Since $\mu(x) = x^{-2}$ we must check if the solution $x \equiv 0$ was lost. The function $x \equiv 0$ is a solution to the original equation, but is not given by the above implicit solution for any choice of C . Hence,

$$\frac{y^2}{2} - \frac{y}{x} + 3x = C \quad \text{and} \quad x \equiv 0$$

give a general solution.

10. We compute the partial derivatives of $M(x, y) = 2y^2 + 2y + 4x^2$ and $N(x, y) = 2xy + x$.

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} (2y^2 + 2y + 4x^2) = 4y + 2, \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x} (2xy + x) = 2y + 1.$$

Although the equation is not exact ($\partial M/\partial y \neq \partial N/\partial x$), the quotient

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{(4y + 2) - (2y + 1)}{2xy + x} = \frac{2y + 1}{x(2y + 1)} = \frac{1}{x}$$

depends on x only, and so the equation has an integrating factor, which can be found by applying formula (8). Namely,

$$\mu(x) = \exp \left(\int \frac{1}{x} dx \right) = \exp (\ln |x|) = |x|.$$

Note that if μ is an integrating factor, then $-\mu$ is an integrating factor as well. This observation allows us to take $\mu(x) = x$. Multiplying the given differential equation by x yields an exact equation

$$(2y^2 + 2y + 4x^2) x dx + x^2 (2y + 1) dy = 0.$$

Therefore,

$$\begin{aligned} F(x, y) &= \int x^2(2y + 1) dy = x^2(y^2 + y) + h(x) \\ \Rightarrow \quad \frac{\partial F}{\partial x} &= 2x(y^2 + y) + h'(x) = (2y^2 + 2y + 4x^2)x \\ \Rightarrow \quad h'(x) &= 4x^3 \quad \Rightarrow \quad h(x) = \int 4x^3 dx = x^4 \\ \Rightarrow \quad F(x, y) &= x^2(y^2 + y) + x^4 = x^2y^2 + x^2y + x^4, \end{aligned}$$

and $x^2y^2 + x^2y + x^4 = C$ is a general solution.

12. Here, $M(x, y) = 2xy^3 + 1$, $N(x, y) = 3x^2y^2 - y^{-1}$. Since

$$\frac{\partial M}{\partial y} = 6xy^2 = \frac{\partial N}{\partial x},$$

the equation is exact. So, we find that

$$\begin{aligned} F(x, y) &= \int (2xy^3 + 1) dx = x^2y^3 + x + g(y), \\ \frac{\partial F}{\partial y} &= 3x^2y^2 + g'(y) = 3x^2y^2 - y^{-1} \\ \Rightarrow \quad g'(y) &= -y^{-1} \quad \Rightarrow \quad g(y) = -\ln|y|, \end{aligned}$$

and the given equation has a general solution

$$x^2y^3 + x - \ln|y| = C.$$

14. Multiplying the given equation by $x^n y^m$ yields

$$(12x^n y^m + 5x^{n+1} y^{m+1}) dx + (6x^{n+1} y^{m-1} + 3x^{n+2} y^m) dy = 0.$$

Therefore,

$$\begin{aligned} \frac{\partial M}{\partial y} &= 12mx^n y^{m-1} + 5(m+1)x^{n+1} y^m, \\ \frac{\partial N}{\partial x} &= 6(n+1)x^n y^{m-1} + 3(n+2)x^{n+1} y^m. \end{aligned}$$

Matching the coefficients, we get a system

$$\begin{cases} 12m = 6(n+1) \\ 5(m+1) = 3(n+2) \end{cases}$$

to determine n and m . This system has the solution $n = 3$, $m = 2$. Thus, the given equation multiplied by $x^3 y^2$, that is,

$$(12x^3 y^2 + 5x^4 y^3) dx + (6x^4 y + 3x^5 y^2) dy = 0,$$

is exact. We compute

$$\begin{aligned} F(x, y) &= \int (12x^3 y^2 + 5x^4 y^3) dx = 3x^4 y^2 + x^5 y^3 + g(y), \\ \frac{\partial F}{\partial y} &= 6x^4 y + 3x^5 y^2 + g'(y) = 6x^4 y + 3x^5 y^2 \\ \Rightarrow \quad g'(y) &= 0 \quad \Rightarrow \quad g(y) = 0, \end{aligned}$$

and so $3x^4 y^2 + x^5 y^3 = C$ is a general solution to the given equation.

16. (a) An equation $Mdx + Ndy = 0$ has an integrating factor $\mu(x + y)$ if and only if the equation

$$\mu(x + y)M(x, y)dx + \mu(x + y)N(x, y)dy = 0$$

is exact. According to Theorem 2, Section 2.4, this means that

$$\frac{\partial}{\partial y} [\mu(x + y)M(x, y)] = \frac{\partial}{\partial x} [\mu(x + y)N(x, y)] .$$

Applying the product and chain rules yields

$$\mu'(x + y)M(x, y) + \mu(x + y)\frac{\partial M(x, y)}{\partial y} = \mu'(x + y)N(x, y) + \mu(x + y)\frac{\partial N(x, y)}{\partial x} .$$

Collecting similar terms yields

$$\begin{aligned} \mu'(x + y) [M(x, y) - N(x, y)] &= \mu(x + y) \left[\frac{\partial N(x, y)}{\partial x} - \frac{\partial M(x, y)}{\partial y} \right] \\ \Leftrightarrow \quad \frac{\partial N/\partial x - \partial M/\partial y}{M - N} &= \frac{\mu'(x + y)}{\mu(x + y)} . \end{aligned} \quad (2.4)$$

The right-hand side of (2.4) depends on $x + y$ only so the left-hand side does.

To find an integrating factor, we let $s = x + y$ and denote

$$G(s) = \frac{\partial N/\partial x - \partial M/\partial y}{M - N} .$$

Then (2.4) implies that

$$\begin{aligned} \frac{\mu'(s)}{\mu(s)} = G(s) &\Rightarrow \ln |\mu(s)| = \int G(s) ds \\ \Rightarrow |\mu(s)| = \exp \left[\int G(s) ds \right] &\Rightarrow \mu(s) = \pm \exp \left[\int G(s) ds \right] . \end{aligned} \quad (2.5)$$

In this formula, we can choose either sign and any integration constant.

- (b) We compute

$$\frac{\partial N/\partial x - \partial M/\partial y}{M - N} = \frac{(1 + y) - (1 + x)}{(3 + y + xy) - (3 + x + xy)} = 1 .$$

Applying formula (2.5), we obtain

$$\mu(s) = \exp \left[\int (1) ds \right] = e^s \quad \Rightarrow \quad \mu(x + y) = e^{x+y} ,$$

Multiplying the given equation by $\mu(x + y)$, we get an exact equation

$$e^{x+y}(3 + y + xy)dx + e^{x+y}(3 + x + xy)dy = 0$$

and follow the procedure of solving exact equations, Section 2.4.

$$\begin{aligned} F(x, y) &= \int e^{x+y}(3+y+xy) dx = e^y \left[(3+y) \int e^x dx + y \int x e^x dx \right] \\ &= e^y [(3+y)e^x + y(x-1)e^x] + h(y) = e^{x+y}(3+xy) + h(y). \end{aligned}$$

Taking the partial derivative of F with respect to y , we find $h(y)$.

$$\begin{aligned} \frac{\partial F}{\partial y} &= e^{x+y}(3+xy+x) + h'(y) = e^{x+y}N(x, y) = e^{x+y}(3+x+xy) \\ \Rightarrow h'(y) &= 0 \quad \Rightarrow \quad h(y) = 0. \end{aligned}$$

Thus, a general solution is

$$e^{x+y}(3+xy) = C.$$

18. The given condition, $xM(x, y) + yN(x, y) \equiv 0$, is equivalent to $yN(x, y) \equiv -xM(x, y)$.

In particular, substituting $x = 0$, we obtain

$$yN(0, y) \equiv -(0)M(0, y) \equiv 0.$$

This implies that $x \equiv 0$ is a solution to the given equation.

To obtain other solutions, we multiply the equation by $x^{-1}y$. This gives

$$\begin{aligned} x^{-1}yM(x, y)dx + x^{-1}yN(x, y)dy &= x^{-1}yM(x, y)dx - x^{-1}xM(x, y)dy \\ &= xM(x, y)(x^{-2}ydx - x^{-1}dy) = -xM(x, y)d(x^{-1}y) = 0. \end{aligned}$$

Therefore, $x^{-1}y = C$ or $y = Cx$.

Thus, a general solution is

$$y = Cx \quad \text{and} \quad x \equiv 0.$$

20. For the equation

$$e^{\int P(x)dx} [P(x)y - Q(x)] dx + e^{\int P(x)dx} dy = 0,$$

we compute

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(e^{\int P(x)dx} [P(x)y - Q(x)] \right) = e^{\int P(x)dx} P(x), \\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(e^{\int P(x)dx} \right) = e^{\int P(x)dx} \frac{d}{dx} \left(\int P(x)dx \right) = e^{\int P(x)dx} P(x). \end{aligned}$$

Therefore, $\partial M/\partial y = \partial N/\partial x$, and the equation is exact.

EXERCISES 2.6: Substitutions and Transformations

2. We can write the equation in the form

$$\frac{dy}{dx} = (y - 4x - 1)^2 = [(y - 4x) - 1]^2 = G(y - 4x),$$

where $G(t) = (t - 1)^2$. Thus, it is of the form $dy/dx = G(ax + by)$.

4. In this equation, the variables are x and t . Its coefficients, $t + x + 2$ and $3t - x - 6$, are linear functions of x and t . Therefore, the given equation is an equation with linear coefficients.
6. The given differential equation is not homogeneous due to the e^{-2x} terms. Since it can be written in the form $dy/dx + P(x)y = Q(x)y^n$, namely,

$$\frac{dy}{dx} - y = e^{2x}y^3.$$

it is a Bernoulli equation. The differential equation does not have linear coefficients, and it is not of the form $y' = G(ax + by)$ either.

8. Here, the variables are y and θ . Writing

$$\frac{dy}{d\theta} = -\frac{y^3 - \theta y^2}{2\theta^2 y} = -\frac{(y/\theta)^3 - (y/\theta)^2}{2(y/\theta)},$$

we see that the right-hand side is a function of y/θ alone. Hence, the equation is homogeneous.

10. First, we write the equation in the form

$$\frac{dy}{dx} = \frac{-3x^2 + y^2}{xy - x^3y^{-1}} = \frac{y^3 - 3x^2y}{xy^2 - x^3} = \frac{(y/x)^3 - 3(y/x)}{(y/x)^2 - 1}.$$

Therefore, it is homogeneous, and we make a substitution $y/x = u$ or $y = xu$. Then $y' = u + xu'$, and the equation becomes

$$u + x \frac{du}{dx} = \frac{u^3 - 3u}{u^2 - 1}.$$

Separating variables and integrating yields

$$x \frac{du}{dx} = \frac{u^3 - 3u}{u^2 - 1} - u = -\frac{2u}{u^2 - 1} \quad \Rightarrow \quad \frac{u^2 - 1}{u} du = -\frac{2}{x} dx$$

$$\begin{aligned}
\Rightarrow \quad \int \frac{u^2 - 1}{u} du &= - \int \frac{2}{x} dx &\Rightarrow \quad \int \left(u - \frac{1}{u} \right) du &= -2 \int \frac{dx}{x} \\
\Rightarrow \quad \frac{1}{2} u^2 - \ln |u| &= -2 \ln |x| + C_1 &\Rightarrow \quad u^2 - \ln(u^2) + \ln(x^4) &= C.
\end{aligned}$$

Substituting back y/x for u and simplifying, we finally get

$$\left(\frac{y}{x}\right)^2 - \ln\left(\frac{y^2}{x^2}\right) + \ln(x^4) = C \quad \Rightarrow \quad \frac{y^2}{x^2} - \ln\left(\frac{y^2}{x^6}\right) = C.$$

12. From

$$\frac{dy}{dx} = -\frac{x^2 + y^2}{2xy} = -\frac{1}{2} \left(\frac{x}{y} + \frac{y}{x} \right),$$

making the substitution $v = y/x$, we obtain

$$\begin{aligned}
v + x \frac{dv}{dx} &= -\frac{1}{2} \left(\frac{1}{v} + v \right) = -\frac{1 + v^2}{2v} &\Rightarrow \quad x \frac{dv}{dx} &= -\frac{1 + v^2}{2v} - v = -\frac{1 + 3v^2}{2v} \\
\Rightarrow \quad \frac{2v dv}{1 + 3v^2} &= -\frac{dx}{x} &\Rightarrow \quad \int \frac{2v dv}{1 + 3v^2} &= - \int \frac{dx}{x} \\
\Rightarrow \quad \frac{1}{3} \ln(1 + 3v^2) &= -\ln|x| + C_2 &\Rightarrow \quad 1 + 3v^2 &= C_1 |x|^{-3},
\end{aligned}$$

where $C_1 = e^{3C_2}$ is any positive constant. Making the back substitution, we finally get

$$\begin{aligned}
1 + 3 \left(\frac{y}{x} \right)^2 &= \frac{C_1}{|x|^3} &\Rightarrow \quad 3 \left(\frac{y}{x} \right)^2 &= \frac{C_1}{|x|^3} - 1 = \frac{C_1 - |x|^3}{|x|^3} \\
\Rightarrow \quad 3|x|y^2 &= C_1 - |x|^3 &\Rightarrow \quad 3|x|y^2 + |x|^3 &= C_1 \Rightarrow 3xy^2 + x^3 = C,
\end{aligned}$$

where $C = \pm C_1$ is any nonzero constant.

14. Substituting $v = y/\theta$ yields

$$\begin{aligned}
v + \theta \frac{dv}{d\theta} &= \sec v + v &\Rightarrow \quad \theta \frac{dv}{d\theta} &= \sec v \\
\Rightarrow \quad \cos v dv &= \frac{d\theta}{\theta} &\Rightarrow \quad \int \cos v dv &= \int \frac{d\theta}{\theta} \\
\Rightarrow \quad \sin v &= \ln |\theta| + C &\Rightarrow \quad y &= \theta \arcsin(\ln |\theta| + C).
\end{aligned}$$

16. We rewrite the equation in the form

$$\frac{dy}{dx} = \frac{y}{x} \left(\ln \frac{y}{x} + 1 \right)$$

and substitute $v = y/x$ to get

$$v + x \frac{dv}{dx} = v (\ln v + 1) \quad \Rightarrow \quad x \frac{dv}{dx} = v \ln v \quad \Rightarrow \quad \int \frac{dv}{v \ln v} = \int \frac{dx}{x}$$

$$\Rightarrow \ln |\ln v| = \ln |x| + C_1 \quad \Rightarrow \quad \ln v = \pm e^{C_1} x = Cx \quad \Rightarrow \quad v = e^{Cx},$$

where $C \neq 0$ is any constant. Note that, separating variables, we lost a solution, $v \equiv 1$, which can be included in the above formula by letting $C = 0$. Thus we have $v = e^{Cx}$, where C is any constant. Substituting back $y = xv$ yields a general solution

$$y = xe^{Cx}$$

to the given equation.

18. With $z = x + y + 2$ and $z' = 1 + y'$, we have

$$\begin{aligned} \frac{dz}{dx} = z^2 + 1 &\Rightarrow \frac{dz}{z^2 + 1} = dx \quad \Rightarrow \quad \int \frac{dz}{z^2 + 1} = \int dx \\ \Rightarrow \arctan z = x + C &\Rightarrow x + y + 2 = z = \tan(x + C) \\ \Rightarrow y = \tan(x + C) - x - 2. \end{aligned}$$

20. Substitution $z = x - y$ yields

$$\begin{aligned} 1 - \frac{dz}{dx} = \sin z &\Rightarrow \frac{dz}{dx} = 1 - \sin z \quad \Rightarrow \quad \frac{dz}{1 - \sin z} = dx \\ \Rightarrow \int \frac{dz}{1 - \sin z} = \int dx = x + C. \end{aligned}$$

The left-hand side integral can be found as follows.

$$\begin{aligned} \int \frac{dz}{1 - \sin z} &= \int \frac{(1 + \sin z)dz}{1 - \sin^2 z} = \int \frac{(1 + \sin z)dz}{\cos^2 z} \\ &= \int \sec^2 z + \int \tan z \sec z dz = \tan z + \sec z. \end{aligned}$$

Thus, a general solution is given implicitly by

$$\tan(x - y) + \sec(x - y) = x + C.$$

22. Dividing the equation by y^3 yields

$$y^{-3} \frac{dy}{dx} - y^{-2} = e^{2x}.$$

We now make a substitution $v = y^{-2}$ so that $v' = -2y^{-3}y'$, and get

$$\frac{dv}{dx} + 2v = -2e^{2x}.$$

This is a linear equation. So,

$$\begin{aligned}\mu(x) &= \exp\left(\int 2dx\right) = e^{2x}, \\ v(x) &= e^{-2x} \int (-2e^{2x}) e^{2x} dx = -(1/2)e^{-2x} (e^{4x} + C) = -\frac{e^{2x} + Ce^{-2x}}{2}.\end{aligned}$$

Therefore,

$$\frac{1}{y^2} = -\frac{e^{2x} + Ce^{-2x}}{2} \quad \Rightarrow \quad y = \pm \sqrt{-\frac{2}{e^{2x} + Ce^{-2x}}}.$$

Dividing the equation by y^3 , we lost a constant solution $y \equiv 0$.

24. We divide this Bernoulli equation by $y^{1/2}$ and make a substitution $v = y^{1/2}$.

$$\begin{aligned}y^{-1/2} \frac{dy}{dx} + \frac{1}{x-2} y^{1/2} &= 5(x-2) \\ \Rightarrow \quad 2 \frac{dv}{dx} + \frac{1}{x-2} v &= 5(x-2) \quad \Rightarrow \quad \frac{dv}{dx} + \frac{1}{2(x-2)} v = \frac{5(x-2)}{2}.\end{aligned}$$

An integrating factor for this linear equation is

$$\mu(x) = \exp\left[\int \frac{dx}{2(x-2)}\right] = \sqrt{|x-2|}.$$

Therefore,

$$\begin{aligned}v(x) &= \frac{1}{\sqrt{|x-2|}} \int \frac{5(x-2)\sqrt{|x-2|}}{2} dx \\ &= \frac{1}{\sqrt{|x-2|}} (|x-2|^{5/2} + C) = (x-2)^2 + C|x-2|^{-1/2}.\end{aligned}$$

Since $y = v^2$, we finally get

$$y = [(x-2)^2 + C|x-2|^{-1/2}]^2.$$

In addition, $y \equiv 0$ is a (lost) solution.

26. Multiplying the equation by y^2 , we get

$$y^2 \frac{dy}{dx} + y^3 = e^x.$$

With $v = y^3$, $v' = 3y^2 y'$, the equation becomes

$$\frac{1}{3} \frac{dv}{dx} + v = e^x \quad \Rightarrow \quad \frac{dv}{dx} + 3v = 3e^x \quad \Rightarrow \quad \frac{d}{dx} (e^{3x} v) = 3e^{4x}$$

$$\Rightarrow v = e^{-3x} \int 3e^{4x} dx = \frac{3e^x}{4} + Ce^{-3x}.$$

Therefore,

$$y = \sqrt[3]{v} = \sqrt[3]{\frac{3e^x}{4} + Ce^{-3x}}.$$

- 28.** First, we note that $y \equiv 0$ is a solution, which will be lost when we divide the equation by y^3 and make a substitution $v = y^{-2}$ to get a linear equation

$$y^{-3} \frac{dy}{dx} + y^{-2} + x = 0 \quad \Rightarrow \quad \frac{dv}{dx} - 2v = 2x.$$

An integrating factor for this equation is

$$\mu(x) = \exp \left[\int (-2) dx \right] = e^{-2x}.$$

Thus,

$$\begin{aligned} v &= e^{2x} \int 2xe^{-2x} dx = e^{2x} \left(-xe^{-2x} + \int e^{-2x} dx \right) \\ &= e^{2x} \left(-xe^{-2x} - \frac{e^{-2x}}{2} + C \right) = -x - \frac{1}{2} + Ce^{2x}. \end{aligned}$$

So,

$$y = \pm v^{-1/2} = \pm \frac{1}{\sqrt{-x - \frac{1}{2} + Ce^{2x}}}.$$

- 30.** We make a substitution $x = u + h$, $y = v + k$, where h and k satisfy system (14) in the text:

$$-4h - k - 1 = 0$$

$$h + k + 3 = 0.$$

Solving yields $h = \frac{2}{3}$, $k = -\frac{11}{3}$. Thus, $x = u + \frac{2}{3}$, $y = v - \frac{11}{3}$. Since $dx = du$, $dy = dv$, the substitution leads to

$$(-4u - v)du + (u + v)dv = 0 \quad \Rightarrow \quad \frac{dv}{du} = \frac{4u + v}{u + v}.$$

Therefore with $z = \frac{v}{u}$, $v' = z + uz'$ we have

$$\begin{aligned} z + uz' &= \frac{4 + z}{1 + z} \text{ or } u \frac{dz}{du} = \frac{4 + z}{1 + z} - z = \frac{4 - z^2}{1 + z} \\ \Rightarrow \quad \frac{1 + z}{4 - z^2} dz &= \frac{du}{u} \end{aligned}$$

$$\begin{aligned}
&\Rightarrow -\frac{3}{4} \log |2 - z| - \frac{1}{4} \log |z + 2| = \log |u| + C_1 \\
&\Rightarrow |u| |2 - z|^{3/4} |z + 2|^{1/4} = e^{-C_1} = C_2 \\
&\Rightarrow \left| x - \frac{2}{3} \right| \left| 2 - \frac{y + \frac{11}{3}}{x - \frac{2}{3}} \right|^{2/3} \left| \frac{y + \frac{11}{3}}{x - \frac{2}{3}} + 2 \right|^{1/4} = C_2
\end{aligned}$$

where C_1 and C_2 are constants. In dividing by $4 - z^2$, we lost the constant solutions $z \equiv 2$ and $z \equiv -2$, but these are included in our answer if we let $C_2 = 0$.

- 32.** We make a substitution $x = u + h$, $y = v + k$, where h and k satisfy system (14) in the text:

$$2h - k + 4 = 0$$

$$h - 2k - 2 = 0.$$

Solving yields $h = -\frac{10}{3}$, $k = -\frac{8}{3}$. Thus, $x = u - \frac{10}{3}$, $y = v - \frac{8}{3}$. Since $dx = du$, $dy = dv$, the substitution leads to

$$(2u - v)du + (u - 2v)dv = 0 \quad \Rightarrow \quad \frac{dv}{du} = \frac{-v + 2u}{2v - u}.$$

Therefore with $z = \frac{v}{u}$, $v' = z + uz'$ we have

$$\begin{aligned}
z + uz' &= \frac{-z + 2}{2z - 1} \text{ or } u \frac{dz}{du} = \frac{-z + 2}{2z - 1} - z = \frac{-2z^2 + 2}{2z - 1} \\
&\Rightarrow \frac{2z - 1}{z^2 - 1} dz = -2 \frac{du}{u} \\
&\Rightarrow \frac{1}{2} \log |1 - z| + \frac{3}{2} \log |z + 1| = -2 \log |u| + C_1 \\
&\Rightarrow |1 - z|^{1/2} |z + 1|^{3/2} u^2 = \left| 1 - \frac{y + \frac{8}{3}}{x + \frac{10}{3}} \right|^{1/2} \left| 1 + \frac{y + \frac{8}{3}}{x + \frac{10}{3}} \right|^{3/2} \left(x + \frac{10}{3} \right)^2 = e^{C_1} = C_2,
\end{aligned}$$

where C_1 and C_2 are constants. In dividing by $z^2 - 1$, we lost the constant solutions $z \equiv 1$ and $z \equiv -1$, but these are included in our answer if we let $C_2 = 0$.

- 34.** In Problem 2, we found that the given equation is of the form $dy/dx = G(y - 4x)$ with $G(u) = (u - 1)^2$. Thus we make a substitution $u = y - 4x$ to get

$$\begin{aligned}
\frac{dy}{dx} &= (y - 4x - 1)^2 \quad \Rightarrow \quad 4 + \frac{du}{dx} = (u - 1)^2 \\
&\Rightarrow \frac{du}{dx} = (u - 1)^2 - 4 = (u - 3)(u + 1) \quad \Rightarrow \quad \int \frac{du}{(u - 3)(u + 1)} = \int dx.
\end{aligned}$$

To integrate the left-hand side, we use partial fractions decomposition,

$$\frac{1}{(u-3)(u+1)} = \frac{1}{4} \left(\frac{1}{u-3} - \frac{1}{u+1} \right).$$

Thus,

$$\begin{aligned} \frac{1}{4} (\ln |u-3| - \ln |u+1|) &= x + C_1 &\Rightarrow \ln \left| \frac{u-3}{u+1} \right| &= 4x + C_2 \\ \Rightarrow \frac{u-3}{u+1} &= Ce^{4x} &\Rightarrow u &= \frac{Ce^{4x} + 3}{1 - Ce^{4x}} \\ \Rightarrow y &= 4x + \frac{Ce^{4x} + 3}{1 - Ce^{4x}}, \end{aligned} \quad (2.6)$$

where $C \neq 0$ is an arbitrary constant. Separating variables, we lost the constant solutions $u \equiv 3$ and $u \equiv -1$, that is, $y = 4x + 3$ and $y = 4x - 1$. While $y = 4x + 3$ can be obtained from (2.6) by setting $C = 0$, the solution $y = 4x - 1$ is not included in (2.6). Therefore, a general solution to the given equation is

$$y = 4x + \frac{Ce^{4x} + 3}{1 - Ce^{4x}} \quad \text{and} \quad y = 4x - 1.$$

- 36.** This equation has linear coefficients. Thus we make a substitution $t = u + h$ and $x = v + k$ with h and k satisfying

$$\begin{aligned} h + k + 2 &= 0, \\ 3h - k - 6 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} h &= 1, \\ k &= -3. \end{aligned}$$

As $dt = du$ and $dx = dv$, the substitution yields

$$(u+v)dv + (3u-v)du = 0 \quad \Rightarrow \quad \frac{du}{dv} = -\frac{u+v}{3u-v} = -\frac{(u/v)+1}{3(u/v)-1}.$$

With $z = u/v$, we have $u = vz$, $u' = z + vz'$, and the equation becomes

$$\begin{aligned} z + v \frac{dz}{dv} &= -\frac{z+1}{3z-1} &\Rightarrow v \frac{dz}{dv} &= -\frac{3z^2+1}{3z-1} \\ \Rightarrow \frac{3z-1}{3z^2+1} dz &= -\frac{1}{v} dv &\Rightarrow \int \frac{3z-1}{3z^2+1} dz &= -\int \frac{1}{v} dv \\ \Rightarrow \int \frac{3z dz}{3z^2+1} - \int \frac{dz}{3z^2+1} &= -\ln |v| + C_1 \\ \Rightarrow \frac{1}{2} \ln (3z^2+1) - \frac{1}{\sqrt{3}} \arctan (z\sqrt{3}) &= -\ln |v| + C_1 \\ \Rightarrow \ln [(3z^2+1)v^2] - \frac{2}{\sqrt{3}} \arctan (z\sqrt{3}) &= C_2. \end{aligned}$$

Making back substitution, after some algebra we get

$$\ln [(x+3)^2 + 3(t-1)^2] + \frac{2}{\sqrt{3}} \arctan \left[\frac{x+3}{\sqrt{3}(t-1)} \right] = C.$$

38. In Problem 6, we have written the equation in the form

$$\frac{dy}{dx} - y = e^{2x}y^3 \quad \Rightarrow \quad y^{-3}\frac{dy}{dx} - y^{-2} = e^{2x}.$$

Making a substitution $u = y^{-2}$ (and so $u' = -2y^{-3}y'$) in this Bernoulli equation, we get

$$\begin{aligned} \frac{du}{dx} + 2u &= -2e^{2x} & \Rightarrow & \quad \mu(x) = \exp\left(\int 2dx\right) = e^{2x} \\ \Rightarrow \quad \frac{d(e^{2x}u)}{dx} &= -2e^{2x}e^{2x} = -2e^{4x} & \Rightarrow & \quad e^{2x}u = \int (-2e^{4x}) dx = -\frac{1}{2}e^{4x} + C \\ \Rightarrow \quad u &= -\frac{1}{2}e^{2x} + Ce^{-2x} & \Rightarrow & \quad y^{-2} = -\frac{1}{2}e^{2x} + Ce^{-2x}. \end{aligned}$$

Also, we lost the constant solution $y \equiv 0$ when divided the equation by y^3 .

40. Since the equation is homogeneous, we make a substitution $u = y/\theta$. Thus, we get

$$\begin{aligned} \frac{dy}{d\theta} &= -\frac{(y/\theta)^3 - (y/\theta)^2}{2(y/\theta)} & \Rightarrow & \quad u + \theta \frac{du}{d\theta} = -\frac{u^3 - u^2}{2u} = -\frac{u^2 - u}{2} \\ \Rightarrow \quad \theta \frac{du}{d\theta} &= -\frac{u^2 + u}{2} & \Rightarrow & \quad \frac{2du}{u(u+1)} = -\frac{d\theta}{\theta} \\ \Rightarrow \quad \int \frac{2du}{u(u+1)} &= -\int \frac{d\theta}{\theta} & \Rightarrow & \quad \ln \frac{u^2}{(u+1)^2} = -\ln |\theta| + C_1, \end{aligned}$$

which gives

$$\frac{u^2}{(u+1)^2} = \frac{C}{\theta}, \quad C \neq 0.$$

Back substitution $u = y/\theta$ yields

$$\frac{y^2}{(\theta + y)^2} = \frac{C}{\theta} \quad \Rightarrow \quad \theta y^2 = C(\theta + y)^2, \quad C \neq 0.$$

When $C = 0$, the above formula gives $\theta \equiv 0$ or $y \equiv 0$, which were lost in separating variables. Also, we lost another solution, $u + 1 \equiv 0$ or $y = -\theta$. Thus, the answer is

$$\theta y^2 = C(\theta + y)^2 \quad \text{and} \quad y = -\theta,$$

where C is an arbitrary constant.

42. Suggested substitution, $y = vx^2$ (so that $y' = 2xv + x^2v'$), yields

$$2xv + x^2 \frac{dv}{dx} = 2vx + \cos v \quad \Rightarrow \quad x^2 \frac{dv}{dx} = \cos v.$$

Solving this separable equation, we obtain

$$\frac{dv}{\cos v} = \frac{dx}{x^2} \quad \Rightarrow \quad \ln |\sec v + \tan v| = -x^{-1} + C_1$$

$$\begin{aligned}\Rightarrow \quad \sec v + \tan v &= \pm e^{C_1} e^{-1/x} = C e^{-1/x} \\ \Rightarrow \quad \sec\left(\frac{y}{x^2}\right) + \tan\left(\frac{y}{x^2}\right) &= C e^{-1/x},\end{aligned}$$

where $C = \pm e^{C_1}$ is an arbitrary nonzero constant. With $C = 0$, this formula also includes lost solutions

$$y = \left[\frac{\pi}{2} + (2k+1)\pi \right] x^2, \quad k = 0, \pm 1, \pm 2, \dots$$

So, together with the other set of lost solutions,

$$y = \left(\frac{\pi}{2} + 2k\pi \right) x^2, \quad k = 0, \pm 1, \pm 2, \dots,$$

we get a general solution to the given equation.

44. From

$$\frac{dy}{dx} = -\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2},$$

using that $a_2 = ka_1$ and $b_2 = kb_1$, we obtain

$$\frac{dy}{dx} = -\frac{a_1x + b_1y + c_1}{ka_1x + kb_1y + c_2} = -\frac{a_1x + b_1y + c_1}{k(a_1x + b_1y) + c_2} = G(a_1x + b_1y),$$

where

$$G(t) = -\frac{t + c_1}{kt + c_2}.$$

48. Apply formula (8), page 50 of Section 2.3, to the transformed Bernoulli equation below equation (10) of Section 2.6, page 74. (If $n = 1$, the Bernoulli equation is linear and formula (8) can be applied directly.)

REVIEW PROBLEMS

2. $y = -8x^2 - 4x - 1 + Ce^{4x}$
4. $\frac{x^3}{6} - \frac{4x^2}{5} + \frac{3x}{4} - Cx^{-3}$
6. $y^{-2} = 2 \ln |1 - x^2| + C$ and $y \equiv 0$
8. $y = (Cx^2 - 2x^3)^{-1}$ and $y \equiv 0$
10. $x + y + 2y^{1/2} + \arctan(x + y) = C$
12. $2ye^{2x} + y^3e^x = C$

$$14. x = \frac{t^2(t-1)}{2} + t(t-1) + 3(t-1) \ln |t-1| + C(t-1)$$

$$16. y = \cos x \ln |\cos x| + C \cos x$$

$$18. y = 1 - 2x + \sqrt{2} \tan(\sqrt{2}x + C)$$

$$20. y = \left(C\theta^{-3} - \frac{12\theta^2}{5}\right)^{1/3}$$

$$22. (3y - 2x + 9)(y + x - 2)^4 = C$$

$$24. 2\sqrt{xy} + \sin x - \cos y = C$$

$$26. y = Ce^{-x^2/2}$$

$$28. (y+3)^2 + 2(y+3)(x+2) - (x+2)^2 = C$$

$$30. y = Ce^{4x} - x - \frac{1}{4}$$

$$32. y^2 = x^2 \ln(x^2) + 16x^2$$

$$34. y = x^2 \sin x + \frac{2x^2}{\pi^2}$$

$$36. \sin(2x + y) - \frac{x^3}{3} + e^y = \sin 2 + \frac{2}{3}$$

$$38. y = \left[2 - \left(\frac{1}{4}\right) \arctan\left(\frac{x}{2}\right)\right]^2$$

$$40. y = \frac{8}{1 - 3e^{-4x} - 4x}$$

TABLES

<i>n</i>	<i>x_n</i>	<i>y_n</i>	<i>n</i>	<i>x_n</i>	<i>y_n</i>
1	0.1	1.475	6	0.6	1.353368921
2	0.2	1.4500625	7	0.7	1.330518988
3	0.3	1.425311875	8	0.8	1.308391369
4	0.4	1.400869707	9	0.9	1.287062756
5	0.5	1.376852388	10	1.0	1.266596983

Table 2–A: Euler’s approximations to $y' = x - y$, $y(0) = 0$, on $[0, 1]$ with $h = 0.1$.

FIGURES

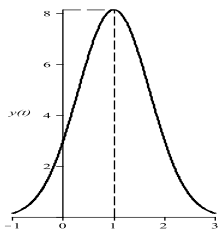


Figure 2–A: The graph of the solution in Problem 28.

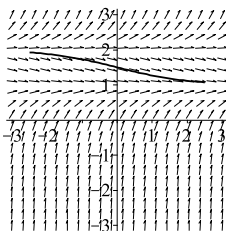


Figure 2–B: The direction field and solution curve in Problem 32.

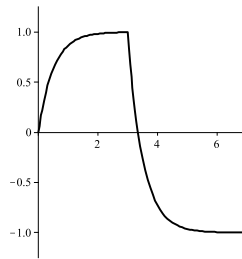


Figure 2–C: The graph of the solution in Problem 32.

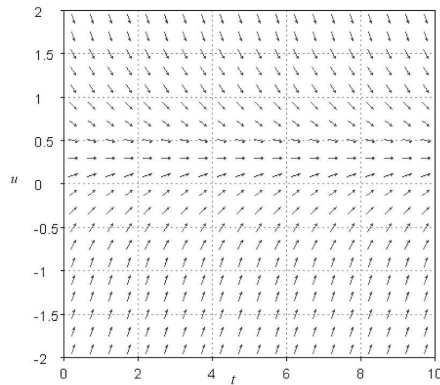


Figure 2–D: The direction field for Problem 40 ($\alpha = \frac{1}{4}, \beta = \frac{1}{2}$).

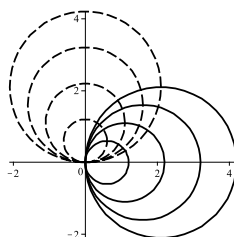


Figure 2–E: Curves and their orthogonal trajectories in Problem 34.