

***Imperfections in Crystalline Solids 1st edition Cai
Solutions Manual***

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Chapter 2

Stress, Strain and Isotropic Elasticity

2.7 Exercise Problems

2.1 A strip of metal is subjected to a uniform tension stress $\sigma_{yy} = \sigma$, as shown in Fig. 2.16.

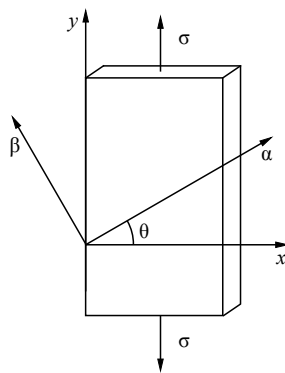


Figure 2.16: A strip of metal subjected to tensile stress.

- (a) Calculate the shear stress $\sigma_{\alpha\beta}$ acting on a plane that is 45° to the horizontal axis.
- (b) Write a general expression for the shear stress $\sigma_{\alpha\beta}$ on a plane making an arbitrary angle θ with the horizontal axis.
- (c) Show that $\sigma_{\alpha\beta}$ is maximum at $\theta = 45^\circ$.
- (d) Calculate the hydrostatic pressure p in the strip.

Solution

(a) $\sigma_{\alpha\beta} = (\cos 45^\circ)(\sin 45^\circ) \sigma_{yy} = \frac{1}{2} \sigma$.

(b) $\sigma_{\alpha\beta} = (\cos \theta)(\sin \theta) \sigma_{yy} = \frac{1}{2} \sin(2\theta) \sigma$.

(c) Maximum of $\sin(2\theta)$ occurs at $\theta = 45^\circ$ where $\sin(2\theta) = \sin 90^\circ = 1$. Therefore, maximum of $\sigma_{\alpha\beta}$ occurs at $\theta = 45^\circ$ where $\sigma_{\alpha\beta} = \frac{1}{2} \sigma$.

(d) $p = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = -\frac{1}{3}\sigma$.

2.2 An FCC single crystal thin film with the orientation shown in Fig. 2.17 is subjected to an equal-biaxial tensile stress σ . Plastic deformation of this crystal will occur by slip on the $(1\bar{1}1)$ plane (shaded) and along the $[011]$ direction as shown. Write an expression for the shear stress acting on the slip plane in the slip direction.

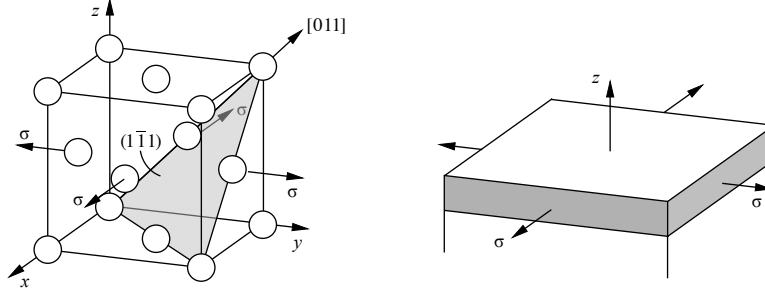


Figure 2.17: An FCC single crystal thin film under biaxial tension.

Solution

In $(x y z)$ coordinate system,

$$\sigma_{xx} = \sigma_{yy} = \sigma, \quad \sigma_{zz} = 0$$

Choose a new coordinate system $(x' y' z')$ in which the x' -, y' -, and z' -axes are along the $[011]$, $[1\bar{1}1]$, and $[21\bar{1}]$ directions respectively. The stress component in question is $\sigma_{x'y'}$.

$$\begin{aligned} \sigma_{x'y'} &= Q_{1k} Q_{2l} \sigma_{kl} \\ &= Q_{11} Q_{21} \sigma_{xx} + Q_{12} Q_{22} \sigma_{yy} \end{aligned}$$

where $Q_{ik} = (\hat{e}'_i \cdot \hat{e}_k)$.

$$\begin{aligned} Q_{11} &= \frac{1}{\sqrt{2}}[011] \cdot [100] = 0 \\ Q_{21} &= \frac{1}{\sqrt{3}}[1\bar{1}1] \cdot [100] = \frac{1}{\sqrt{3}} \\ Q_{12} &= \frac{1}{\sqrt{2}}[011] \cdot [010] = \frac{1}{\sqrt{2}} \\ Q_{22} &= \frac{1}{\sqrt{3}}[1\bar{1}1] \cdot [010] = -\frac{1}{\sqrt{3}} \end{aligned}$$

Therefore,

$$\sigma_{x'y'} = \left(\frac{1}{\sqrt{2}}\right) \left(-\frac{1}{\sqrt{3}}\right) \sigma = -\frac{1}{\sqrt{6}} \sigma$$

2.3 A single crystal of BCC iron is pulled in uniaxial tension along a cube direction, $[100]$. We assume slip occurs on the $(\bar{2}11)$ plane and in the $[111]$ direction as shown in Fig. 2.18(a). The

critical resolved shear stress (CRSS) for slip to occur is 10 MPa. Calculate the tensile stress at which plastic yielding begins.

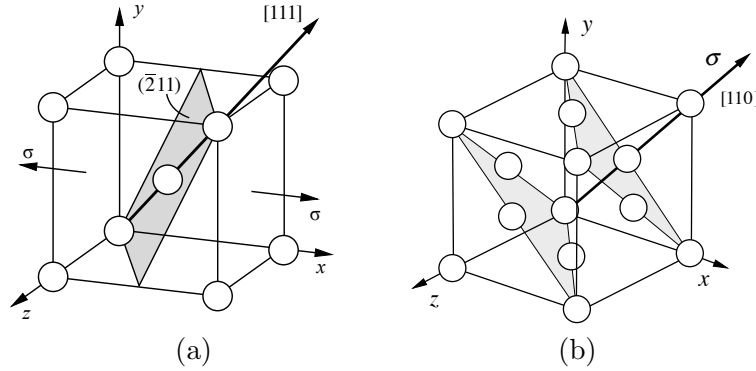


Figure 2.18: (a) A single crystal of BCC iron subjected to tension along the $[1\ 0\ 0]$ direction. (b) An FCC single crystal subjected to tension along the $[1\ 1\ 0]$ direction. Two $(1\ 1\ \bar{1})$ planes are shaded to help show the atomic positions.

Solution

In $(x\ y\ z)$ coordinate system,

$$\sigma_{xx} = \sigma, \quad \sigma_{zz} = \sigma_{yy} = 0$$

Choose a new coordinate system $(x'\ y'\ z')$ in which the x' -, y' -, and z' -axes are along the $[1\ 1\ 1]$, $[\bar{2}\ 1\ 1]$, and $[0\ \bar{1}\ 1]$ directions respectively. The stress component for the CRSS is $\sigma_{x'y'}$.

$$\begin{aligned} \sigma_{x'y'} &= Q_{1k} Q_{2l} \sigma_{kl} \\ &= Q_{11} Q_{21} \sigma_{xx} \end{aligned}$$

where $Q_{ik} = (\hat{e}'_i \cdot \hat{e}_k)$.

$$\begin{aligned} Q_{11} &= \frac{1}{\sqrt{3}} [1\ 1\ 1] \cdot [1\ 0\ 0] = \frac{1}{\sqrt{3}} \\ Q_{21} &= \frac{1}{\sqrt{6}} [\bar{2}\ 1\ 1] \cdot [1\ 0\ 0] = -\frac{2}{\sqrt{6}} \end{aligned}$$

Therefore,

$$\begin{aligned} \sigma_{x'y'} &= \left(\frac{1}{\sqrt{3}} \right) \left(-\frac{2}{\sqrt{6}} \right) \sigma = -\frac{\sqrt{2}}{3} \sigma \\ |\sigma_{x'y'}| &= 10\text{ MPa} \sigma = \frac{3}{\sqrt{2}} \times 10\text{ MPa} = 21.2\text{ MPa} \end{aligned}$$

2.4 An FCC single crystal is subjected to a uniform tensile stress of magnitude σ in the $[1\ 1\ 0]$ direction, as shown in Fig. 2.18(b). Calculate all the non-zero stress components in the cubic (xyz) axes.

Solution

Let the x, y, z axes in the (xyz) coordinate system be aligned along the $[100]$, $[010]$, and $[001]$ directions, respectively. Choose a new coordinate system $(x'y'z')$ in which the x' -, y' -, and z' -axes are along the $[110]$, $[\bar{1}11]$, and $[1\bar{1}2]$ directions respectively. Here we know that

$$\sigma_{x'x'} = \sigma, \quad \sigma_{y'y'} = \sigma_{z'z'} = 0$$

The stress components in the (xyz) coordinate system are

$$\begin{aligned} \sigma_{ij} &= Q_{ik}^T Q_{jl}^T \sigma'_{kl} \\ &= Q_{i1}^T Q_{j1}^T \sigma_{x'x'} \end{aligned}$$

where $Q_{ik}^T = (\hat{e}'_k \cdot \hat{e}_i)$.

$$\begin{aligned} Q_{11}^T &= \frac{1}{\sqrt{2}} [110] \cdot [100] = \frac{1}{\sqrt{2}} \\ Q_{21}^T &= \frac{1}{\sqrt{2}} [110] \cdot [010] = \frac{1}{\sqrt{2}} \\ Q_{31}^T &= \frac{1}{\sqrt{2}} [110] \cdot [001] = 0 \end{aligned}$$

Therefore,

$$\begin{aligned} \sigma_{xx} &= Q_{11}^T Q_{11}^T \sigma_{x'x'} = \frac{1}{2} \sigma \\ \sigma_{yy} &= Q_{21}^T Q_{21}^T \sigma_{x'x'} = \frac{1}{2} \sigma \\ \sigma_{xy} &= Q_{11}^T Q_{21}^T \sigma_{x'x'} = \frac{1}{2} \sigma \\ \sigma_{xz} &= \sigma_{yz} = \sigma_{zz} = 0 \end{aligned}$$

2.5 Consider a single crystal rod subjected to a uniaxial tensile stress σ , as shown in Fig. 2.19. Plastic deformation would occur if the resolved shear stress τ on a certain crystallographic plane (with normal vector $\hat{n}$) and along a certain crystallographic direction $\hat{m}$ exceeds a threshold value. Express the resolved shear stress τ in terms of the tensile stress σ and the angles θ and ϕ that vectors $\hat{n}$ and $\hat{m}$ make with the tensile axis. The ratio $S = \tau/\sigma$ is called the *Schmid factor*.

Solution

$$\begin{aligned} \tau &= \cos \theta \cos \phi \sigma \\ S &= \cos \theta \cos \phi \end{aligned}$$

2.6 The stress field of an infinite isotropic elastic medium containing a pressurized cylindrical hole (of radius r_0) has the following form,

$$\begin{aligned} \sigma_{rr} &= -\frac{A}{r^2} \\ \sigma_{\theta\theta} &= \frac{A}{r^2} \\ \sigma_{r\theta} &= 0 \end{aligned} \tag{2.81}$$

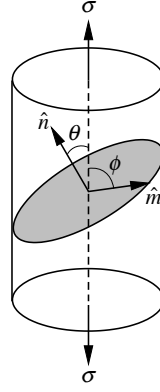


Figure 2.19: Single crystal subjected to uniaxial tension.

for $r \geq r_0$. Transform the stress field into Cartesian components: σ_{xx} , σ_{yy} , σ_{xy} . Make contour plots of σ_{rr} , $\sigma_{r\theta}$, $\sigma_{\theta\theta}$ and σ_{xx} , σ_{yy} , σ_{xy} .

Solution

Let coordinate system A be the cylindrical (r, θ, z) coordinate system and coordinate system A' be the Cartesian (x, y, z) coordinate system. Let matrix Q be,

$$Q_{ik} = (\hat{e}_i^{xyz} \cdot \hat{e}_k^{r\theta z}) = \begin{bmatrix} (\hat{e}_x \cdot \hat{e}_r) & (\hat{e}_x \cdot \hat{e}_\theta) & (\hat{e}_x \cdot \hat{e}_z) \\ (\hat{e}_y \cdot \hat{e}_r) & (\hat{e}_y \cdot \hat{e}_\theta) & (\hat{e}_y \cdot \hat{e}_z) \\ (\hat{e}_z \cdot \hat{e}_r) & (\hat{e}_z \cdot \hat{e}_\theta) & (\hat{e}_z \cdot \hat{e}_z) \end{bmatrix}$$

Given that

$$\begin{aligned} \hat{e}_r &= \hat{e}_x \cos \theta + \hat{e}_y \sin \theta \\ \hat{e}_\theta &= -\hat{e}_x \sin \theta + \hat{e}_y \cos \theta \end{aligned}$$

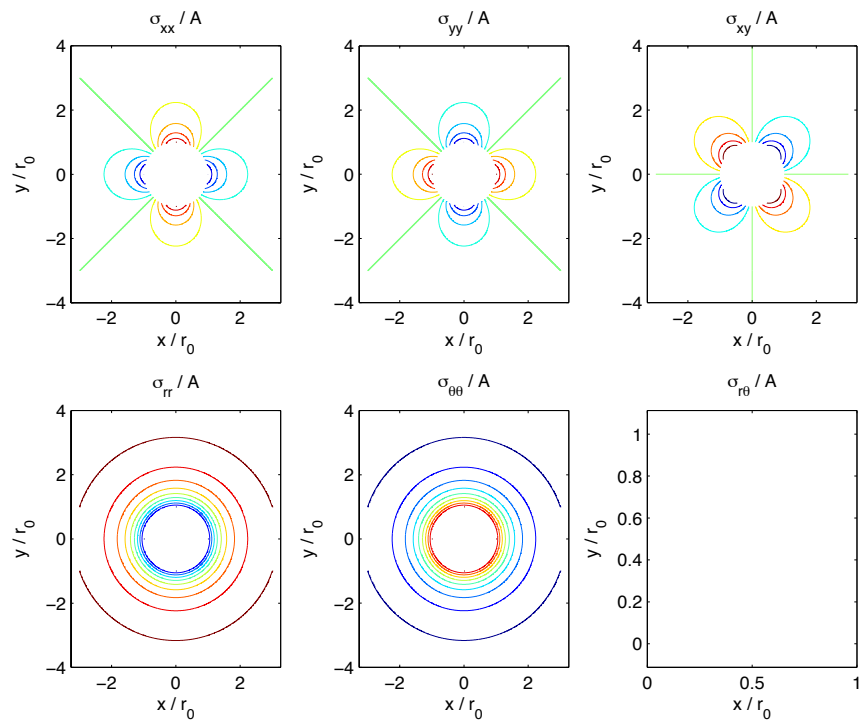
we have,

$$Q_{ik} = \begin{bmatrix} Q_{xr} & Q_{x\theta} & Q_{xz} \\ Q_{yr} & Q_{y\theta} & Q_{yz} \\ Q_{zr} & Q_{z\theta} & Q_{zz} \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \sigma_{xx} &= Q_{xr} Q_{xr} \sigma_{rr} + Q_{x\theta} Q_{x\theta} \sigma_{\theta\theta} = -(\cos^2 \theta - \sin^2 \theta) \left(\frac{A}{r^2} \right) \\ \sigma_{yy} &= Q_{yr} Q_{yr} \sigma_{rr} + Q_{y\theta} Q_{y\theta} \sigma_{\theta\theta} = (\cos^2 \theta - \sin^2 \theta) \left(\frac{A}{r^2} \right) \\ \sigma_{zz} &= Q_{zr} Q_{zr} \sigma_{rr} + Q_{z\theta} Q_{z\theta} \sigma_{\theta\theta} = 0 \\ \sigma_{xy} &= Q_{xr} Q_{yr} \sigma_{rr} + Q_{x\theta} Q_{y\theta} \sigma_{\theta\theta} = -2 \sin \theta \cos \theta \left(\frac{A}{r^2} \right) \\ \sigma_{yz} &= Q_{yr} Q_{zr} \sigma_{rr} + Q_{y\theta} Q_{z\theta} \sigma_{\theta\theta} = 0 \\ \sigma_{xz} &= Q_{xr} Q_{zr} \sigma_{rr} + Q_{x\theta} Q_{z\theta} \sigma_{\theta\theta} = 0 \end{aligned}$$

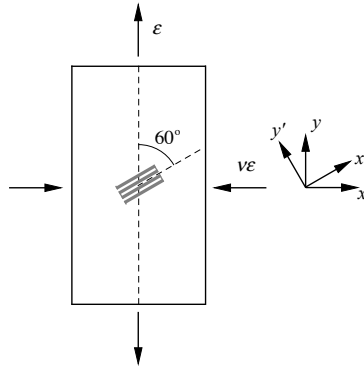
The results can be simplified as: $\sigma_{xx} = -A \cos(2\theta)/r^2$, $\sigma_{yy} = A \cos(2\theta)/r^2$, $\sigma_{xy} = -A \sin(2\theta)/r^2$. They can be written in terms of x , y , z as

$$\begin{aligned}\sigma_{xx} &= -A \frac{x^2 - y^2}{(x^2 + y^2)^2} \\ \sigma_{yy} &= A \frac{x^2 - y^2}{(x^2 + y^2)^2} \\ \sigma_{xy} &= -A \frac{2xy}{(x^2 + y^2)^2}\end{aligned}$$



2.7 A strain gauge measures tensile strain along a given direction through the change of electrical resistance of thin wires aligned in that direction. Consider a solid subjected to uniaxial tensile stress, which induces a normal strain ε in the tensile direction, and a normal strain $-\nu\varepsilon$ in the perpendicular directions, as shown in Fig. 2.20(a). Express the strain measured by a strain gauge oriented at 60° from the tensile axis.

Solution



For the coordinate system $(x\ y)$ shown,

$$\varepsilon_{xx} = -\nu \varepsilon$$

$$\varepsilon_{yy} = \varepsilon$$

The coordinate system $(x'\ y')$ is rotated from the coordinate system $(x\ y)$ by 30° . What the strain gauge measures is component $\varepsilon_{x'x'}$.

$$\begin{aligned}\varepsilon_{x'x'} &= Q_{1k} Q_{1l} \varepsilon_{kl} \\ &= Q_{11} Q_{11} \varepsilon_{xx} + Q_{12} Q_{12} \varepsilon_{yy}\end{aligned}$$

where $Q_{ik} = (\hat{e}'_i \cdot \hat{e}_k)$.

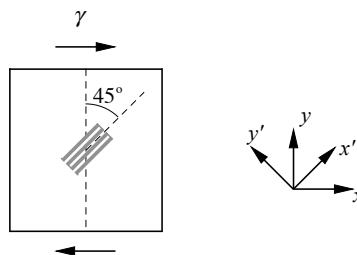
$$\begin{aligned}Q_{11} &= \cos 30^\circ = \frac{\sqrt{3}}{2} \\ Q_{12} &= \cos 60^\circ = \frac{1}{2}\end{aligned}$$

Therefore,

$$\begin{aligned}\varepsilon_{x'x'} &= \frac{\sqrt{3}}{2} \frac{\sqrt{3}}{2} (-\nu \varepsilon) + \frac{1}{2} \frac{1}{2} \varepsilon \\ &= \frac{1 - 3\nu}{4} \varepsilon\end{aligned}$$

2.8 Express the strain measured by a strain gauge oriented at 45° from the vertical axis of a solid subjected to an engineering shear strain of γ , as shown in Fig. 2.20(b).

Solution



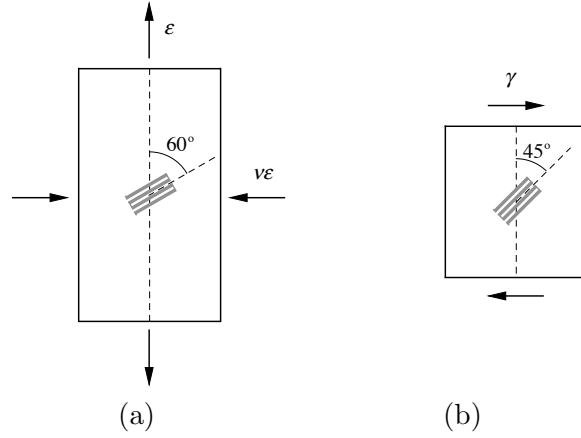


Figure 2.20: Strain gauge.

For the coordinate system $(x\ y)$ shown,

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{\gamma}{2}$$

The coordinate system $(x'\ y')$ is rotated from the coordinate system $(x\ y)$ by 45° . What the strain gauge measures is component $\varepsilon_{x'x'}$.

$$\begin{aligned}\varepsilon_{x'x'} &= Q_{1k} Q_{1l} \varepsilon_{kl} \\ &= Q_{11} Q_{12} \varepsilon_{xy} + Q_{12} Q_{11} \varepsilon_{yx}\end{aligned}$$

where $Q_{ik} = (\hat{e}'_i \cdot \hat{e}_k)$.

$$\begin{aligned}Q_{11} &= \cos 45^\circ = \frac{\sqrt{2}}{2} \\ Q_{12} &= \cos 45^\circ = \frac{\sqrt{2}}{2}\end{aligned}$$

Therefore,

$$\begin{aligned}\varepsilon_{x'x'} &= \frac{\sqrt{2}}{2} \frac{\sqrt{2}}{2} \frac{\gamma}{2} + \frac{\sqrt{2}}{2} \frac{\sqrt{2}}{2} \frac{\gamma}{2} \\ &= \frac{\gamma}{2}\end{aligned}$$

2.9 A thin film is subjected to strain $\varepsilon_{xx} = \varepsilon_1$, $\varepsilon_{yy} = \varepsilon_2$, $\varepsilon_{zz} = \varepsilon_3$. Write an expression for the axial strain ε in the arbitrary direction shown in Fig. 2.21.

Solution

Choose a coordinate system $(x'\ y'\ z')$ such that the x' -axis is along the arbitrary direction shown in Fig. 2.21.

$$\begin{aligned}\varepsilon &= \varepsilon_{x'x'} = Q_{1k} Q_{1l} \varepsilon_{kl} \\ &= Q_{11} Q_{11} \varepsilon_{xx} + Q_{12} Q_{12} \varepsilon_{yy} + Q_{13} Q_{13} \varepsilon_{zz}\end{aligned}$$

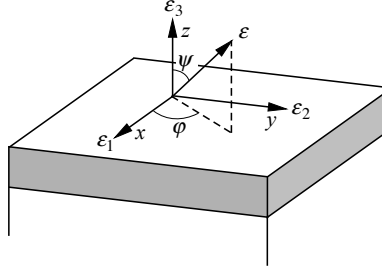


Figure 2.21: A thin film subjected to strain ε_1 , ε_2 , ε_3 in the x , y , z directions, respectively.

where $Q_{ik} = (\hat{e}'_i \cdot \hat{e}_k)$.

$$Q_{11} = \sin \psi \cos \varphi$$

$$Q_{12} = \sin \psi \sin \varphi$$

$$Q_{13} = \cos \psi$$

Therefore,

$$\varepsilon = (\sin \psi \cos \varphi)^2 \varepsilon_1 + (\sin \psi \sin \varphi)^2 \varepsilon_2 + (\cos \psi)^2 \varepsilon_3$$

2.10 A solid cube with shear modulus μ and Poisson's ratio ν is subjected to a compressive stress along z , i.e. $\sigma_{zz} = -p$, but is not allowed to expand in the x direction due to constraints imposed by rigid plates. The solid is allowed to expand freely in the y direction. Obtain the normal stress and strain of the solid in x , y and z directions.

Solution

$$\varepsilon_{xx} = \frac{1}{E} [\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})]$$

$$\varepsilon_{yy} = \frac{1}{E} [\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})]$$

$$\varepsilon_{zz} = \frac{1}{E} [\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})]$$

For this problem, $\varepsilon_{xx} = 0$ and $\sigma_{yy} = 0$. Therefore,

$$0 = \varepsilon_{xx} = \frac{1}{E} (\sigma_{xx} - \nu \sigma_{zz})$$

$$\sigma_{xx} = \nu \sigma_{zz} = -\nu p$$

In summary,

$$\begin{aligned}
 \sigma_{xx} &= -\nu p \\
 \sigma_{yy} &= 0 \\
 \sigma_{zz} &= -p \\
 \epsilon_{xx} &= -\nu p \\
 \epsilon_{yy} &= -\frac{\nu}{E}(\sigma_{xx} + \sigma_{zz}) = \frac{\nu(1+\nu)p}{E} \\
 \epsilon_{zz} &= \frac{1}{E}(\sigma_{zz} - \nu\sigma_{xx}) = -\frac{(1-\nu^2)p}{E}
 \end{aligned}$$

2.11 Repeat the analysis in Problem 2.10 but for a solid constrained (i.e. unable to expand) in both x and y directions.

Solution

$$\begin{aligned}
 \epsilon_{xx} &= \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] \\
 \epsilon_{yy} &= \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})] \\
 \epsilon_{zz} &= \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})]
 \end{aligned}$$

For this problem, $\epsilon_{xx} = 0$ and $\epsilon_{yy} = 0$. By symmetry, we have $\sigma_{xx} = \sigma_{yy}$. Therefore,

$$\begin{aligned}
 0 = \epsilon_{xx} &= \frac{1}{E}[(1-\nu)\sigma_{xx} - \nu\sigma_{zz}] \\
 \sigma_{xx} = \sigma_{yy} &= \frac{\nu}{1-\nu}\sigma_{zz} = -\frac{\nu}{1-\nu}p
 \end{aligned}$$

In summary,

$$\begin{aligned}
 \sigma_{xx} &= -\frac{\nu}{1-\nu}p \\
 \sigma_{yy} &= -\frac{\nu}{1-\nu}p \\
 \sigma_{zz} &= -p \\
 \epsilon_{xx} &= 0 \\
 \epsilon_{yy} &= 0 \\
 \epsilon_{zz} &= \frac{1}{E}(\sigma_{zz} - 2\nu\sigma_{xx}) = -\frac{p}{E}\left(1 - \frac{2\nu^2}{1-\nu}\right)
 \end{aligned}$$

2.12 Estimate the compressive stress that arises in a block of aluminum when the temperature increases from 0°C to 30°C if its volume is not allowed to expand. Use the thermoelastic properties of aluminum given in Table B.1. If the block were allowed to expand freely, it would experience a thermal strain of $\epsilon^T = \alpha \Delta T$ in all three direction with zero stress, where α is the thermal expansion coefficient and ΔT is the temperature increase.

Solution

$$\begin{aligned}\varepsilon_{xx}^{\text{tot}} &= \varepsilon_{xx}^{\text{el}} + \varepsilon_{xx}^{\text{T}} = \varepsilon_{xx}^{\text{el}} + \alpha \Delta T = 0 \\ \varepsilon_{yy}^{\text{tot}} &= \varepsilon_{yy}^{\text{el}} + \varepsilon_{yy}^{\text{T}} = \varepsilon_{yy}^{\text{el}} + \alpha \Delta T = 0 \\ \varepsilon_{zz}^{\text{tot}} &= \varepsilon_{zz}^{\text{el}} + \varepsilon_{zz}^{\text{T}} = \varepsilon_{zz}^{\text{el}} + \alpha \Delta T = 0\end{aligned}$$

Therefore, $\varepsilon_{xx}^{\text{el}} = \varepsilon_{yy}^{\text{el}} = \varepsilon_{zz}^{\text{el}} = -\alpha \Delta T$.

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = B \cdot (\varepsilon_{xx}^{\text{el}} + \varepsilon_{yy}^{\text{el}} + \varepsilon_{zz}^{\text{el}}) = -3B \alpha \Delta T$$

From Table B.1, we find $B = 75.9 \text{ GPa}$, $\alpha = 23.1 \times 10^{-6} \text{ K}^{-1}$ for aluminum. Therefore, for $\Delta T = 30 \text{ K}$, we have

$$\begin{aligned}\sigma_{xx} = \sigma_{yy} = \sigma_{zz} &= -3B \alpha \Delta T \\ &= -3 \times 75.9 \times 23.1 \times 10^{-6} \times 30 \text{ GPa} \\ &= 0.158 \text{ GPa} \\ &= 158 \text{ MPa}\end{aligned}$$

2.13 Consider an infinite elastic solid containing a cylindrical hole of radius r_0 , which is filled with a gas with pressure p . The stress field in the solid is given in Eq. (2.81). Express the coefficient A in terms of the pressure p inside the hole and the hole radius r_0 . Verify that this stress field satisfies the equilibrium condition, Eq. (2.56).

Solution

$$\begin{aligned}-p &= \sigma_{rr}|_{r=r_0} = -\frac{A}{r_0^2} \\ A &= p r_0^2\end{aligned}$$

Because $\sigma_{r\theta} = \sigma_{\theta z} = \sigma_{rz} = 0$, and all the remaining stress components are independent of θ and z , it can be easily seen that, ($F_r = F_\theta = F_z = 0$)

$$\begin{aligned}\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r} \sigma_{r\theta} &= 0 \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r} \sigma_{rz} &= 0\end{aligned}$$

The only equilibrium condition we need to verify is

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r} (\sigma_{rr} - \sigma_{\theta\theta}) = 0$$

Because $\sigma_{rr} = -A/r^2$ and $\sigma_{\theta\theta} = A/r^2$,

$$\begin{aligned}\frac{\partial \sigma_{rr}}{\partial r} &= 2 \frac{A}{r^3} \\ \frac{1}{r} (\sigma_{rr} - \sigma_{\theta\theta}) &= \frac{1}{r} \left(-\frac{A}{r^2} - \frac{A}{r^2} \right) = -2 \frac{A}{r^3}\end{aligned}$$

Therefore,

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} (\sigma_{rr} - \sigma_{\theta\theta}) = 0$$

Because $\sigma_{r\theta} = \sigma_{rz} = 0$, we have

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r} (\sigma_{rr} - \sigma_{\theta\theta}) = 0$$

2.14 Given the stress field expression, Eq. (2.81), for an infinite elastic solid containing a pressurized cylindrical hole,

- (a) Find the corresponding strain field. Express the strain field in Cartesian (xyz) components.
- (b) Verify that the strain field satisfies the compatibility condition Eq. (2.67).
- (c) Verify that the displacement field $u_r = C/r$, $u_\theta = 0$ is consistent with the strain field, and express the coefficient C in terms of hole radius r_0 and pressure p .

Assume the solid is a linear isotropic material with shear modulus μ and Poisson's ratio ν .

Solution

(a) We have the plane strain condition $\varepsilon_{zz} = 0$. To find σ_{zz} , we use the condition that

$$\begin{aligned} 0 = \varepsilon_{zz} &= \frac{1}{E} [\sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta})] \\ \sigma_{zz} &= \nu(\sigma_{rr} + \sigma_{\theta\theta}) \end{aligned}$$

Given that $\sigma_{rr} = -A/r^2$ and $\sigma_{\theta\theta} = A/r^2$, $A = p r_0^2$ (see Problem 2.13), we have

$$\sigma_{zz} = 0$$

So we also have a plane stress condition in the solid (outside the hole).

$$\begin{aligned} \varepsilon_{rr} &= \frac{1}{E} [\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})] \\ &= -\frac{1+\nu}{E} \frac{A}{r^2} = -\frac{A}{2\mu} \frac{1}{r^2} \\ \varepsilon_{\theta\theta} &= \frac{1}{E} [\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})] \\ &= \frac{1+\nu}{E} \frac{A}{r^2} = \frac{A}{2\mu} \frac{1}{r^2} \\ \varepsilon_{r\theta} &= \varepsilon_{rz} = \varepsilon_{\theta z} = 0 \end{aligned}$$

Similar to the stress transformation rule from cylindrical to Cartesian coordinates derived in Problem 2.6, we have similar expressions for the strain transformation, i.e.

$$Q_{ik} = \begin{bmatrix} Q_{xr} & Q_{x\theta} & Q_{xz} \\ Q_{yr} & Q_{y\theta} & Q_{yz} \\ Q_{zr} & Q_{z\theta} & Q_{zz} \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
\varepsilon_{xx} &= Q_{xr} Q_{xr} \varepsilon_{rr} + Q_{x\theta} Q_{x\theta} \varepsilon_{\theta\theta} = -(\cos^2 \theta - \sin^2 \theta) \left(\frac{A}{2\mu r^2} \right) \\
\varepsilon_{yy} &= Q_{yr} Q_{yr} \varepsilon_{rr} + Q_{y\theta} Q_{y\theta} \varepsilon_{\theta\theta} = (\cos^2 \theta - \sin^2 \theta) \left(\frac{A}{2\mu r^2} \right) \\
\varepsilon_{zz} &= 0 \\
\varepsilon_{xy} &= Q_{xr} Q_{yr} \varepsilon_{rr} + Q_{x\theta} Q_{y\theta} \varepsilon_{\theta\theta} = -2 \sin \theta \cos \theta \left(\frac{A}{2\mu r^2} \right) \\
\varepsilon_{yz} &= Q_{yr} Q_{zr} \varepsilon_{rr} + Q_{y\theta} Q_{z\theta} \varepsilon_{\theta\theta} = 0 \\
\varepsilon_{xz} &= Q_{xr} Q_{zr} \varepsilon_{rr} + Q_{x\theta} Q_{z\theta} \varepsilon_{\theta\theta} = 0
\end{aligned}$$

The results can be simplified as: $\sigma_{xx} = -A \cos(2\theta)/r^2$, $\sigma_{yy} = A \cos(2\theta)/r^2$, $\sigma_{xy} = -A \sin(2\theta)/r^2$. They can be written in terms of x , y , z as

$$\begin{aligned}
\varepsilon_{xx} &= -\frac{A}{2\mu} \frac{x^2 - y^2}{(x^2 + y^2)^2} \\
\varepsilon_{yy} &= \frac{A}{2\mu} \frac{x^2 - y^2}{(x^2 + y^2)^2} \\
\varepsilon_{xy} &= -\frac{A}{2\mu} \frac{2xy}{(x^2 + y^2)^2}
\end{aligned}$$

(b) The compatibility condition to be verified is

$$\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - 2\varepsilon_{xy,xy} = 0$$

$$\begin{aligned}
\varepsilon_{xx,yy} &= \frac{\partial^2}{\partial y^2} \varepsilon_{xx} = \frac{A}{2\mu} \frac{6(x^4 - 6xy + y^4)}{(x^2 + y^2)^4} \\
\varepsilon_{yy,xx} &= \frac{\partial^2}{\partial x^2} \varepsilon_{yy} = \frac{A}{2\mu} \frac{6(x^4 - 6xy + y^4)}{(x^2 + y^2)^4} \\
\varepsilon_{xy,xy} &= \frac{\partial^2}{\partial x \partial y} \varepsilon_{xy} = \frac{A}{2\mu} \frac{6(x^4 - 6xy + y^4)}{(x^2 + y^2)^4}
\end{aligned}$$

Therefore,

$$\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - 2\varepsilon_{xy,xy} = 0$$

(c) Suppose $u_r = C/r$ and $u_\theta = u_z = 0$, then

$$\begin{aligned}\varepsilon_{rr} &= \frac{\partial u_r}{\partial r} = -\frac{C}{r^2} \\ \varepsilon_{\theta\theta} &= \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = \frac{C}{r^2} \\ \varepsilon_{zz} &= \frac{\partial u_z}{\partial z} = 0 \\ \varepsilon_{r\theta} &= \frac{1}{2} \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) = 0 \\ \varepsilon_{rz} &= \frac{1}{2} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) = 0 \\ \varepsilon_{\theta z} &= \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right) = 0\end{aligned}$$

These are consistent with the displacement fields in (a) as long as

$$C = \frac{A}{2\mu} = \frac{p r_0^2}{2\mu}$$

2.15 Given the stress and strain expressions in Problems 2.13 and 2.14 for an infinite solid containing a pressurized cylindrical hole, express the total elastic energy per unit length along the hole in terms of the pressure p and hole radius r_0 using the following two approaches and show that they agree with each other.

- (a) Integrate the strain energy density Δw over the entire volume of the solid.
- (b) Integrate the work done on the interior surface of the cylindrical hole as the pressure gradually increases from 0 to p .

Solution

(a)

$$\begin{aligned}\Delta w &= \frac{1}{2} \sigma_{rr} \varepsilon_{rr} + \frac{1}{2} \sigma_{\theta\theta} \varepsilon_{\theta\theta} = \frac{1}{2} \frac{A^2}{2\mu r^4} + \frac{1}{2} \frac{A^2}{2\mu r^4} \\ &= \frac{A^2}{2\mu r^4} \\ E &= \int_{r_0}^{\infty} r dr \int_0^{2\pi} d\theta \Delta w = \int_{r_0}^{\infty} dr 2\pi r \frac{A^2}{2\mu r^4} \\ &= \frac{\pi A^2}{2\mu r_0^2} = \frac{\pi (p r_0^2)^2}{2\mu r_0^2} = \frac{\pi p^2 r_0^2}{2\mu}\end{aligned}$$

(b)

$$E = \frac{1}{2} p (2\pi r_0) u_r|_{r=r_0}$$

From Problem 2.14, we have $u_r = C/r$ with $C = p r_0^2 / (2\mu)$, so that

$$u_r|_{r=r_0} = \frac{p r_0^2}{2\mu} \frac{1}{r_0}$$

$$E = \frac{1}{2} p (2\pi r_0) \left(\frac{p r_0^2}{2\mu} \frac{1}{r_0} \right) = \frac{\pi p^2 r_0^2}{2\mu}$$

consistent with the result in (a).

2.16 Find the expressions for the displacement field for Kelvin's problem, in which a unit force in the z direction is applied at the origin in an infinite elastic medium. Express the displacement field in cylindrical coordinates. Which strain and stress components are non-zero in cylindrical coordinates?

Solution

In Cartesian coordinates, the displacement field caused by a point force in the z -direction is,

$$u_x = \frac{1}{16\pi\mu(1-\nu)} \frac{xz}{(x^2 + y^2 + z^2)^{3/2}}$$

$$u_y = \frac{1}{16\pi\mu(1-\nu)} \frac{yz}{(x^2 + y^2 + z^2)^{3/2}}$$

$$u_z = \frac{1}{16\pi\mu(1-\nu)} \left[\frac{3-4\nu}{(x^2 + y^2 + z^2)^{1/2}} + \frac{z^2}{(x^2 + y^2 + z^2)^{3/2}} \right]$$

In cylindrical coordinates (r, θ, z) , where $r = \sqrt{x^2 + y^2}$, the displacement field is,

$$u_r = \frac{1}{16\pi\mu(1-\nu)} \frac{rz}{(r^2 + z^2)^{3/2}}$$

$$u_\theta = 0$$

$$u_z = \frac{1}{16\pi\mu(1-\nu)} \left[\frac{3-4\nu}{(r^2 + z^2)^{1/2}} + \frac{z^2}{(r^2 + z^2)^{3/2}} \right]$$

In cylindrical coordinates, the strain field is related to the displacement field through,

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\varepsilon_{r\theta} = \frac{1}{2} \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right)$$

$$\varepsilon_{rz} = \frac{1}{2} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right)$$

$$\varepsilon_{\theta z} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right)$$

Given that u_r and u_z are independent of θ and $u_\theta = 0$, we have $\varepsilon_{r\theta} = \varepsilon_{\theta z} = 0$. The non-zero strain components are: ε_{rr} , $\varepsilon_{\theta\theta}$, ε_{zz} , and ε_{rz} .

From generalized Hooke's law, we have $\sigma_{r\theta} = \sigma_{\theta z} = 0$. The non-zero stress components are: σ_{rr} , $\sigma_{\theta\theta}$, σ_{zz} , and σ_{rz} .