

***Introduction to Chemical Engineering
Fluid Mechanics 1st edition Deen
Solutions Manual***

SKU: 9781107123779-SOLUTIONS

Solution: Chapter 2

2.1 Cavitation

(a) Given:

$$\begin{aligned} Q &= 3.93 \times 10^{-3} \text{ m}^3/\text{s} & h_2 - h_1 &= 3 \text{ m} \\ D &= 5 \text{ cm} & \rho &= 1.0 \times 10^3 \text{ kg/m}^3 \\ L &= 50 \text{ m} & \nu &= 1.0 \times 10^{-6} \text{ m}^2/\text{s} \end{aligned}$$

Find: $|\Delta\mathcal{P}|$ from hose inlet to pump intake.

Calculate Re and f :

$$\begin{aligned} U &= \frac{4Q}{\pi D^2} = \frac{4(3.93 \times 10^{-3})}{\pi(5 \times 10^{-2})^2} = 2.0 \text{ m/s} \\ \text{Re} &= \frac{UD}{\nu} = \frac{(2)(0.05)}{1 \times 10^{-6}} = 1.0 \times 10^5 \end{aligned}$$

Flow is turbulent and Blasius eq. [Eq. (2.2-9)] is applicable.

$$f = 0.0791 \text{ Re}^{-1/4} = 0.0791(1 \times 10^5)^{-1/4} = 4.45 \times 10^{-3}$$

Find $|\Delta\mathcal{P}|$ from Eq. (2.3-6):

$$\begin{aligned} |\Delta\mathcal{P}| &= \frac{2\rho U^2 L}{D} f = \frac{2(1000)(2)^2 50}{0.05} (4.45 \times 10^{-3}) \\ &= 3.56 \times 10^4 \text{ Pa} \end{aligned}$$

(b) Find P_2 (pump intake) for $P_1 = 1.01 \times 10^5 \text{ Pa}$ (nearly empty pool).

$$\begin{aligned} \Delta\mathcal{P} &= -|\Delta\mathcal{P}| = P_2 - P_1 + \rho g(h_2 - h_1) \\ P_2 &= P_1 - |\Delta\mathcal{P}| - \rho g(h_2 - h_1) \\ &= 1.01 \times 10^5 - 3.56 \times 10^4 - (1000)(9.81)(3) \\ &= 3.60 \times 10^4 \text{ Pa} \end{aligned}$$

Although below atmospheric, $P_2 > P_v = 2.34 \times 10^3 \text{ Pa}$, so cavitation will be absent.

(c) Maximum Q without cavitation.

P_1 is lowest when the pool is nearly empty, so P_2 is also lowest then. $Q = Q_{\max}$ when $P_2 = P_v$ and $P_1 = 1.01 \times 10^5 \text{ Pa}$.

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Find $|\Delta\mathcal{P}|$ at $Q = Q_{\max}$:

$$\begin{aligned} |\Delta\mathcal{P}| &= P_1 - P_2 - \rho g(h_2 - h_1) \\ &= 1.01 \times 10^5 - 2.34 \times 10^3 - (1000)(9.81)(3) \\ &= 6.92 \times 10^4 \text{ Pa} \end{aligned}$$

Re is now unknown, but $\text{Re}\sqrt{f}$ can be found and the Prandtl-Kármán eq. used [Eq. (2.2-7)].

$$\begin{aligned} f &= \frac{D}{2\rho U^2} \frac{|\Delta\mathcal{P}|}{L}, \text{Re} = \frac{UD}{\nu} \\ \text{Re}\sqrt{f} &= \frac{D}{\nu} \left(\frac{D|\Delta\mathcal{P}|}{2\rho L} \right)^{1/2} \\ &= \frac{0.05}{1 \times 10^{-6}} \left[\frac{0.05(6.92 \times 10^4)}{2(1000)(50)} \right]^{1/2} = 9.30 \times 10^3 \\ f &= [4.0 \log(\text{Re}\sqrt{f}) - 0.4]^{-2} = 4.18 \times 10^{-3} \\ Q_{\max} &= \frac{\pi D^2}{4} U_{\max} = \frac{\pi D^2}{4} \left(\frac{D}{2\rho f} \frac{|\Delta\mathcal{P}|}{L} \right)^{1/2} \\ &= \frac{\pi(0.05)^2}{4} \left[\frac{0.05(6.92 \times 10^4)}{2(1000)(4.18 \times 10^{-3})(50)} \right]^{1/2} \\ &= 5.65 \times 10^{-3} \text{ m}^3/\text{s} \end{aligned}$$

2.2 Bottling Honey

Given: Gravity feed with $|\Delta h| = 2 \text{ m}$.

Tube length $= L = 4 \text{ m}$.

Honey properties as in Table 1.2.

Mass flow rate $\geq 17 \text{ oz/min}$.

Find: Minimum tube diameter D .

Volume flow rate:

$$\begin{aligned} 17 \text{ oz} \left(\frac{1 \text{ lb}}{16 \text{ oz}} \right) \left(\frac{0.454 \text{ kg}}{\text{lb}} \right) &= 0.482 \text{ kg} \\ w = \text{mass flow rate} &= \left(0.482 \frac{\text{kg}}{\text{min}} \right) \left(\frac{1 \text{ min}}{60 \text{ s}} \right) = 8.04 \times 10^{-3} \text{ kg/s} \\ Q = \text{vol. flow rate} &= \frac{w}{\rho} = \frac{8.04 \times 10^{-3}}{1.42 \times 10^3} = 5.66 \times 10^{-6} \text{ m}^3/\text{s} \end{aligned}$$

The high viscosity suggests laminar flow. Poiseuille's equation:

$$\begin{aligned} Q &= \frac{\pi D^4}{128 \mu} \frac{|\Delta\mathcal{P}|}{L} \\ D^4 &= \frac{128 \mu L Q}{\pi |\Delta\mathcal{P}|}, \quad D = \left(\frac{128 \mu L Q}{\pi |\Delta\mathcal{P}|} \right)^{1/4} \\ |\Delta\mathcal{P}| &= |\Delta P| + \rho g |\Delta h| = \rho g |\Delta h| \quad (\text{atm } P \text{ at both ends}) \end{aligned}$$

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$$D = \left(\frac{128\nu L Q}{\pi g |\Delta h|} \right)^{1/4} = \left[\frac{128(1.34 \times 10^{-2})(4)(5.66 \times 10^{-6})}{\pi(9.81)(2)} \right]^{1/4}$$
$$= 2.82 \times 10^{-2} \text{ m} = 2.82 \text{ cm.}$$

This is a very reasonable diameter, indicating that the bottling process is feasible.

Check Re to confirm that the flow is really laminar:

$$U = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \frac{(5.66 \times 10^{-6})}{(2.82 \times 10^{-2})^2} = 9.08 \times 10^{-3} \text{ m/s}$$
$$\text{Re} = \frac{UD}{\nu} = \frac{(9.08 \times 10^{-3})(2.82 \times 10^{-2})}{1.34 \times 10^{-2}} = 1.91 \times 10^{-2}$$

This is much smaller than 2100, which confirms the applicability of Poiseuille's equation.

2.3 Filling a Boiler

Given: $V = 7.57 \text{ m}^3$ (volume to be transferred)

Copper tubing with $D = 2.60 \text{ cm}$ and $L = 15 \text{ m}$.

$$\Delta h = 3 \text{ m}$$

$$|\Delta P| = 4.0 \times 10^5 \text{ Pa}$$

(a) Filling time at 20°C ($\rho = 998 \text{ kg/m}^3$, $\nu = 1.00 \times 10^{-6} \text{ m}^2/\text{s}$)

Assume that the flow is turbulent and might be affected by roughness. Although Re is unknown, $\text{Re}\sqrt{f}$ can be found from what is given, making the Colebrook – White eq. [Eq. (2.5.2)] a good choice.

$$f = \frac{D}{2\rho U^2} \frac{|\Delta \mathcal{P}|}{L}, \text{Re} = \frac{UD}{\nu}$$
$$\text{Re}\sqrt{f} = \frac{D}{\nu} \left(\frac{D|\Delta \mathcal{P}|}{2\rho L} \right)^{1/2}$$
$$\Delta \mathcal{P} = \Delta P + \rho g \Delta h = -|\Delta P| + \rho g \Delta h.$$
$$|\Delta \mathcal{P}| = -\Delta \mathcal{P} = |\Delta P| - \rho g \Delta h$$
$$= 4.0 \times 10^5 - (998)(9.81)(3) = 3.71 \times 10^5 \text{ Pa}$$
$$\text{Re}\sqrt{f} = \frac{0.026}{1.00 \times 10^{-6}} \left[\frac{0.026(3.71 \times 10^5)}{2(998)(15)} \right]^{1/2} = 1.48 \times 10^4$$
$$\frac{k}{3.7D} = \frac{1.5 \times 10^{-6}}{3.7(0.026)} = 1.56 \times 10^{-5} \text{ (} k \text{ for "drawn metals", Table 2.2)}$$
$$f = \left[4 \log \left(\frac{1.26}{\text{Re}\sqrt{f}} + \frac{k}{3.7D} \right) \right]^{-2}$$
$$= \left[4 \log \left(\frac{1.26}{1.48 \times 10^4} + 1.56 \times 10^{-5} \right) \right]^{-2}$$
$$= 3.91 \times 10^{-3}$$
$$U = \left(\frac{D}{2\rho f} \frac{|\Delta \mathcal{P}|}{L} \right)^{1/2} = \left[\frac{0.026(3.71 \times 10^5)}{2(998)(3.91 \times 10^{-3})(15)} \right]^{1/2} = 9.07 \frac{\text{m}}{\text{s}}$$

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$$Q = \frac{\pi D^2}{4} U = \frac{\pi (0.026)^2}{4} (9.07) = 4.82 \times 10^{-3} \frac{\text{m}^3}{\text{s}}$$
$$t = \frac{V}{Q} = \frac{7.57}{4.82 \times 10^{-3}} = 1.57 \times 10^3 \text{ s} \left(\frac{1 \text{ min}}{60 \text{ s}} \right) = 26.2 \text{ min}$$

Check Re to confirm turbulent flow:

$$\text{Re} = \frac{UD}{\nu} = \frac{9.07(0.026)}{1.00 \times 10^{-6}} = 2.36 \times 10^5 > 2100.$$

(b) Filling time at 52°C ($\rho = 987 \text{ kg/m}^3$, $\nu = 5.29 \times 10^{-4}/987 = 0.536 \times 10^{-6} \text{ m}^2/\text{s}$).

$$\text{Re}\sqrt{f} \propto \nu^{-1} \rho^{-1/2}$$

$$\text{Re}\sqrt{f} = 1.48 \times 10^4 \left(\frac{1.00}{0.536} \right) \left(\frac{998}{987} \right)^{1/2} = 2.78 \times 10^4$$

$$\frac{k}{3.7D} = 1.56 \times 10^{-5} \text{ as before}$$

$$f = \left[4 \log \left(\frac{1.26}{2.78 \times 10^4} + 1.56 \times 10^{-5} \right) \right]^{-2} = 3.52 \times 10^{-3}$$

$$U \propto (\rho f)^{-1/2}$$

$$U = 9.07 \left[\frac{998(3.91)}{987(3.52)} \right]^{1/2} = 9.61 \frac{\text{m}}{\text{s}}$$

$$t \propto \frac{1}{U}$$

$$t = 26.2 \left(\frac{9.07}{9.61} \right) = 24.7 \text{ min}$$

Only 1.5 min is saved.

2.4 Syringe Pump

(a) Best syringe size from among 1 ml, 5 ml, and 50 ml.

Total volume to be delivered is

$$V = \left(\frac{100 \mu\text{L}}{\text{min}} \right) (30 \text{ min}) = \left(\frac{0.1 \text{ mL}}{\text{min}} \right) (30 \text{ min}) = 3 \text{ mL}.$$

The 1 ml syringe is too small.

Pusher velocity needed for other syringes:

$$U_s = \frac{4Q}{\pi D_s^2}$$

$$U_s(5 \text{ ml}) = \frac{4(0.1 \text{ cm}^3/\text{min})}{\pi (1.21 \text{ cm})^2} = 8.70 \times 10^{-2} \frac{\text{cm}}{\text{min}} = 0.87 \frac{\text{mm}}{\text{min}}$$

$$U_s(50 \text{ ml}) = \frac{4(0.1 \text{ cm}^3/\text{min})}{\pi (2.67 \text{ cm})^2} = 1.79 \times 10^{-2} \frac{\text{cm}}{\text{min}} = 0.18 \frac{\text{mm}}{\text{min}}$$

Either would work, but 5 ml is preferable because its U_s is not as close to the limits of the pump.

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(b) $|\Delta P|$ in horizontal tubing

$$\begin{aligned}\text{Re} &= \frac{UD}{\nu} = \left(\frac{4Q}{\pi D^2} \right) \frac{D}{\nu} = \frac{4Q}{\pi D \nu} \\ Q &= \left(100 \times 10^{-6} \frac{L}{\text{min}} \right) \left(10^{-3} \frac{\text{m}^3}{L} \right) \left(\frac{1 \text{ min}}{60 \text{ s}} \right) = 1.67 \times 10^{-9} \frac{\text{m}^3}{\text{s}} \\ \text{Re} &= \frac{4(1.67 \times 10^{-9})}{\pi (8.6 \times 10^{-4})(1 \times 10^{-6})} = 2.47 \text{ (laminar flow)} \\ |\Delta P| &= |\Delta \mathcal{P}| = \frac{128 \mu L Q}{\pi D^4} \text{ (Poiseuille)} \\ &= \frac{128(1 \times 10^{-3})(0.5)(1.67 \times 10^{-9})}{\pi (8.6 \times 10^{-4})^4} = 62.2 \text{ Pa}\end{aligned}$$

As shown, $|\Delta P|$ depends only on Q and the tubing dimensions. The syringe size does not matter.

(c) $|\Delta P|$ for outlet above inlet, $h_2 - h_1 = 10 \text{ cm}$.

$|\Delta \mathcal{P}|$ (from Poiseuille eq.) will be the same as before.

$$\begin{aligned}\Delta \mathcal{P} &= \Delta P + \rho g \Delta h \\ -\Delta P &= -\Delta \mathcal{P} + \rho g \Delta h = |\Delta \mathcal{P}| + \rho g \Delta h \quad (\Delta \mathcal{P} < 0) \\ \rho g \Delta h &= (1 \times 10^3)(9.81)(0.1) = 981 \text{ Pa} \\ -\Delta P &= 62 + 981 = 1043 \text{ Pa} = |\Delta P|\end{aligned}$$

Moving the liquid upward requires a larger $|\Delta P|$. Because P_{outlet} is fixed at atmospheric, P_{inlet} must be larger.

2.5 Flue Gases

Given: D , L , and $|\Delta P|$

Find: w (mass flow rate)

Assume that the flow is turbulent, in which case it might be affected by roughness. Although Re is not known initially, $\text{Re}\sqrt{f}$ can be calculated, making the Colebrook – White equation [Eq. (2.5-2)] a good choice.

$$\begin{aligned}\frac{1}{\sqrt{f}} &= -4 \log \left(\frac{1.26}{\text{Re}\sqrt{f}} + \frac{k}{3.7D} \right) \quad \text{Eq. (2.5-2)} \\ \text{Re}\sqrt{f} &= \frac{UD}{\nu} \left(\frac{D|\Delta P|}{2\rho U^2 L} \right)^{1/2} = \left(\frac{D^3 |\Delta P|}{2\nu^2 \rho L} \right)^{1/2} \\ \nu &= \frac{\mu}{\rho} = \frac{2.6 \times 10^{-5}}{0.63} = 4.13 \times 10^{-5} \frac{\text{m}^2}{\text{s}} \\ \text{Re}\sqrt{f} &= \left[\frac{(6.4 \times 10^{-2})^3 (63)}{2(4.13 \times 10^{-5})^2 (0.63)(6.1)} \right]^{1/2} = 1.12 \times 10^3 \\ k &= 4.6 \times 10^{-5} \text{ m for steel (Table 2.2)} \\ \frac{k}{D} &= \frac{4.6 \times 10^{-5}}{6.4 \times 10^{-2}} = 7.19 \times 10^{-4}\end{aligned}$$

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$$\begin{aligned}
 f &= \left[4 \log \left(\frac{1.26}{\text{Re} \sqrt{f}} + \frac{k}{3.7D} \right) \right]^{-2} \\
 &= \left[4 \log \left(\frac{1.26}{1.12 \times 10^3} + \frac{7.19 \times 10^{-4}}{3.7} \right) \right]^{-2} \\
 &= 7.53 \times 10^{-3}
 \end{aligned}$$

From Fig. 2.6, $\text{Re} \cong 10^4$, confirming that the flow is turbulent.

Find U from f :

$$U = \left(\frac{D}{2\rho f} \frac{|\Delta P|}{L} \right)^{1/2} = \left[\frac{(6.4 \times 10^{-2})(63)}{2(0.63)(7.53 \times 10^{-3})(6.1)} \right]^{1/2} = 8.35 \frac{\text{m}}{\text{s}}$$

Check Re :

$$\text{Re} = \frac{UD}{\nu} = \frac{(8.35)(6.4 \times 10^{-2})}{4.13 \times 10^{-5}} = 1.29 \times 10^4 \quad (\text{turbulent, as assumed})$$

Find w :

$$w = \rho Q = \frac{\pi D^2}{4} U \rho = \frac{\pi}{4} (6.4 \times 10^{-2})^2 (8.35)(0.63) = 1.69 \times 10^{-2} \frac{\text{kg}}{\text{s}}$$

2.6 Hydraulic Fracturing

Given Q , D , L , Δh , and P_2 (outlet), find P_1 (inlet). Start with Re :

$$\begin{aligned}
 \text{Re} &= \frac{UD}{\nu} = \left(\frac{4Q}{\pi D^2} \right) \frac{D}{\nu} = \frac{4Q}{\pi \nu D} \\
 \nu &= \frac{0.467 \times 10^{-3}}{0.983 \times 10^3} = 4.75 \times 10^{-7} \frac{\text{m}^2}{\text{s}} \text{ for } 60^\circ\text{C water.} \\
 \text{Re} &= \frac{4}{\pi} \frac{0.21}{(4.75 \times 10^{-7})(0.15)} = 3.75 \times 10^6 \quad (\text{turbulent flow})
 \end{aligned}$$

See if roughness is important:

$$\begin{aligned}
 k &= 4.6 \times 10^{-5} \text{ m for steel (Table 2.2)} \\
 \frac{k}{D} &= \frac{4.6 \times 10^{-5}}{0.15} = 3.1 \times 10^{-4}
 \end{aligned}$$

Inspection of Fig. 2.6 indicates that roughness significantly increases f at this Re and k/D .

For turbulent flow with roughness, Eq. (2.5–3) is convenient for finding f :

$$\begin{aligned}
 f &= \left\{ 3.6 \log \left[\frac{6.9}{\text{Re}} + \left(\frac{k}{3.7D} \right)^{1.11} \right] \right\}^{-2} \\
 &= \left\{ 3.6 \log \left[\frac{6.9}{3.75 \times 10^6} + \left(\frac{3.1 \times 10^{-4}}{3.7} \right)^{1.11} \right] \right\}^{-2} = 3.81 \times 10^{-3}
 \end{aligned}$$

Find $|\Delta \mathcal{P}|$ from f :

$$|\Delta \mathcal{P}| = \frac{2\rho U^2 L f}{D} \quad U = \frac{4Q}{\pi D^2} = 11.9 \frac{\text{m}}{\text{s}}$$

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$$\begin{aligned}
&= \frac{2(0.983 \times 10^3)(11.9)^2(2.3 \times 10^3)(3.81 \times 10^{-3})}{0.15} \\
&= 1.62 \times 10^7 \text{ Pa} = 16.2 \text{ MPa}
\end{aligned}$$

Find P_1 (top of pipe):

$$\begin{aligned}
\Delta \mathcal{P} &= \Delta P + \rho g \Delta h \\
-\Delta P &= P_1 - P_2 = -\Delta \mathcal{P} + \rho g \Delta h = |\Delta \mathcal{P}| - \rho g(\Delta h) \quad (\Delta \mathcal{P} < 0, \Delta h < 0) \\
\rho g|\Delta h| &= (0.983 \times 10^3)(9.81)(2.3 \times 10^3) = 2.22 \times 10^7 \text{ Pa} = 22.2 \text{ MPa} \\
P_1 &= P_2 + |\Delta \mathcal{P}| - \rho g|\Delta h| \\
&= 44 + 16.2 - 22.2 = 38 \text{ MPa}
\end{aligned}$$

Note that even though the downward flow encounters a large resistance in the pipe, $P_1 < P_2$ due to the large $\rho g|\Delta h|$.

2.7 Drag Reduction

Given:

$$\begin{aligned}
U &= 3.0 \text{ m/s} & \rho &= 850 \text{ kg/m}^3 \\
D &= 1.0 \text{ m} & \nu &= 1.06 \times 10^{-5} \text{ m}^2/\text{s} \\
L &= 100 \text{ km}
\end{aligned}$$

Find: $|\Delta P| (= |\Delta \mathcal{P}|)$ for maximum drag reduction.

In Example 2.3–2 it was found that $\text{Re} = 2.83 \times 10^5$ and roughness was assumed (implicitly) to be negligible. Using k for steel in Table 2.2, $k/D = 4.6 \times 10^{-5}$. Figure 2.6 confirms that roughness effects are indeed small at this Re and k/D .

Approach: Find f iteratively from Eq. (P2.7-1)

Find $|\Delta P|$ from f and the given data.

For $\text{Re} = 2.83 \times 10^5$, Fig. P2.7 gives $f \cong 6 \times 10^{-4}$. An improved value from Eq. (P2.7-1) is

$$\begin{aligned}
\text{Re}\sqrt{f} &= (2.83 \times 10^5)(6 \times 10^{-4})^{1/2} = 6.93 \times 10^3 \\
f &= [19 \log(\text{Re}\sqrt{f}) - 32.4]^{-2} = [19 \log(6.93 \times 10^3) - 32.4]^{-2} \\
&= 6.07 \times 10^{-4}
\end{aligned}$$

A second refinement is

$$\begin{aligned}
\text{Re}\sqrt{f} &= (2.83 \times 10^5)(6.07 \times 10^{-4})^{1/2} = 6.97 \times 10^3 \\
f &= [19 \log(6.97 \times 10^3) - 32.4]^{-2} = 6.06 \times 10^{-4}
\end{aligned}$$

which is adequate convergence. Then

$$|\Delta P| = \frac{2\rho U^2 L}{D} f \propto f$$

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$|\Delta P| = |\Delta P|_0 = 5.59 \times 10^6 \text{ Pa}$ and $f = f_0 = 3.66 \times 10^{-3}$ without additives (Example 2.3-2).

$$\begin{aligned} |\Delta P|_{\text{MDR}} &= |\Delta P|_0 \left(\frac{f_{\text{MDR}}}{f_0} \right) = 5.59 \times 10^6 \left(\frac{6.06 \times 10^{-4}}{3.66 \times 10^{-3}} \right) \\ &= 9.26 \times 10^5 \text{ Pa} \end{aligned}$$

This is an 83% reduction. Polymeric additives have been used, for example, in the Trans-Alaska Pipeline.

2.8 Economic Pipe Diameter

(a) Pipe diameter D^* that minimizes total cost

$$C = K_1 D^n L + K_2 Q |\Delta \mathcal{P}|$$

Relate $Q|\Delta \mathcal{P}|$ to D :

$$\begin{aligned} |\Delta \mathcal{P}| &= \frac{2\rho U^2 f L}{D} = \frac{2\rho f L}{D} \left(\frac{4Q}{\pi D^2} \right)^2 = \frac{32}{\pi^2} \frac{\rho f L Q^2}{D^5} \\ f &\cong a \text{Re}^{-b} = a \left(\frac{4}{\pi} \frac{Q}{v D} \right)^{-b} = a \left(\frac{\pi}{4} \right)^b \left(\frac{v D}{Q} \right)^b \\ Q|\Delta \mathcal{P}| &= \frac{32a}{\pi^2} \left(\frac{\pi}{4} \right)^b \frac{\rho L Q^3}{D^5} \frac{v^b D^b}{Q^b} \end{aligned}$$

$$\text{Let } c = \frac{32a}{\pi^2} \left(\frac{\pi}{4} \right)^b$$

$$Q|\Delta \mathcal{P}| = c \rho L v^b Q^{3-b} D^{b-5}$$

Total cost is now

$$C(D) = K_1 D^n L + K_2 c \rho L v^b Q^{3-b} D^{b-5}$$

To minimize C , set $dC/dD = 0$:

$$\begin{aligned} \frac{dC}{dD} &= 0 = nK_1 D^{n-1} L + K_2 c \rho L v^b Q^{3-b} (b-5) D^{b-6} \\ nK_1 D^{n-1} &= (5-b)(K_2 c \rho L v^b Q^{3-b} D^{b-6}) \\ D^{n-1-b+6} &= D^{5+n-b} = \frac{(5-b)K_2 c}{nK_1} \rho v^b Q^{3-b} \\ &= \frac{(5-b)K_2 c}{nK_1} \rho^{1-b} \mu^b Q^{3-b} \\ D^* &= \left[\frac{(5-b)K_2 c}{nK_1} \right]^{1/(5+n-b)} \rho^{(1-b)/(5+n-b)} \mu^{b/(5+n-b)} Q^{(3-b)/(5+n-b)} \end{aligned}$$

(b) Dependence of D^* on ρ and μ

Use $n = 1.5$ and $b = 0.16$.

$$\rho \text{ exponent is } \frac{1-b}{5+n-b} = \frac{0.84}{6.34} = 0.132$$

$$\mu \text{ exponent is } \frac{b}{5+n-b} = \frac{0.16}{6.34} = 0.025$$

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Both are small, implying insensitivity of D^* to property values.

(c) Dependence of U^* on Q

$$\begin{aligned} D^* &\propto Q^{(3-b)/(5+n-b)} = Q^{2.84/6.34} = Q^{0.45} \\ U^* &\propto \frac{Q}{(D^*)^2} \propto \frac{Q}{Q^{2(3-b)/(5+n-b)}} \\ 1 - \frac{2(3-b)}{(5+n-b)} &= \frac{5+n-b-6+2b}{5+n-b} = \frac{-1+n+b}{5+n-b} \\ &= \frac{-1+1.5+0.16}{5+1.5-0.16} = 0.10 \\ U^* &\propto Q^{0.10} \end{aligned}$$

Whereas D^* is somewhat sensitive to Q , U^* is relatively insensitive.

2.9 Microfluidic Device

(a) Flow-rate ratios.

From flow in = flow out, $Q_1 = Q_5$.

From $L_2 = L_4$, $Q_2 = Q_4$

From $L_3 = L_2/2$, $Q_3 = 2Q_2$.

$$\begin{aligned} Q_1 &= Q_2 + Q_3 + Q_4 = Q_2 + 2Q_2 + Q_2 = 4Q_2 \\ \frac{Q_2}{Q_1} &= \frac{1}{4}, \frac{Q_3}{Q_1} = \frac{1}{2}, \frac{Q_4}{Q_1} = \frac{1}{4}, \frac{Q_5}{Q_1} = 1 \end{aligned}$$

(b) Calculate Q_1 = overall flow rate for $P_A - P_D = 3000$ Pa, $L_1 = 1$ mm, and square channels with side length $100 \mu\text{m}$.

Let r_{ij} = resistance between nodes i and j

r_k = resistance of channel k

There are 3 parallel channels between B and C . With $r_i \propto L_i$,

$$\frac{1}{r_{BC}} = \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{1}{r_2} + \frac{2}{r_2} + \frac{1}{r_2} = \frac{4}{r_2}$$

With $L_2 = 4L_1$, $r_2 = 4r_1$ and

$$\frac{1}{r_{BC}} = \frac{1}{r_1} \quad \text{or} \quad r_{BC} = r_1.$$

With $L_5 = L_1$, $r_5 = r_1$. Adding the series resistances,

$$r_{AD} = r_1 + r_{BC} + r_5 = 3r_1$$

Friction factor for laminar flow:

$$f = \frac{D_H}{2\rho U^2} \frac{|\Delta P|}{L} = \frac{c}{\text{Re}_H} \quad c = \text{const.}$$

Solution: Chapter 2

For square channels, $c = 14.23$, $D_H = a =$ side length (Table 2.1 with $a = b$), and $\text{Re}_4 = UD_H/\nu = Ua/\nu$. Thus

$$\frac{a}{2\rho U^2} \frac{|\Delta P|}{L} = \frac{c\nu}{Ua} \quad \text{or} \quad U = \frac{a^2}{2c\mu} \frac{|\Delta P|}{L}$$

Apply to segment 1:

$$|\Delta P|_1 = \frac{|\Delta P|_{AD}}{3} = 1000 \text{ Pa} \quad \text{because } r_1 = r_{AD}/3.$$

$$L_1 = 1 \times 10^{-3} \text{ m}$$

$$a = 1 \times 10^{-4} \text{ m}$$

$$U_1 = \frac{(1 \times 10^{-4})^2}{2(14.23)(1 \times 10^{-3})} \frac{1000}{1 \times 10^{-3}} = 0.351 \frac{\text{m}}{\text{s}}$$

$$Q_1 = U_1 a^2 = (0.351)(1 \times 10^{-4} \text{ m})^2 = 3.51 \times 10^{-9} \frac{\text{m}^3}{\text{s}}$$

2.10 Murray's Law

(a) Radius that minimizes total energy expended (E_t) for one vessel.

$$\begin{aligned} E_t &= Q|\Delta \mathcal{P}| + b\pi R^2 L \\ |\Delta \mathcal{P}| &= \frac{8\mu L Q}{\pi R^4} \quad (\text{Poiseuille}) \\ E_t &= \frac{8\mu L Q^2}{\pi R^4} + b\pi R^2 L \equiv AR^{-4} + BR^2 \end{aligned}$$

Minimize E_t with respect to R , for constant $A = 8\mu L Q^2/\pi$ and $B = 6\pi L$:

$$\begin{aligned} \frac{dE_t}{dR} &= -4AR^{-5} + 2BR = 0 \\ R^6 &= \frac{2A}{B} = \frac{16\mu L Q^2}{\pi} \left(\frac{1}{b\pi L} \right) = \frac{16\mu Q^2}{b\pi^2} \\ R &= \left(\frac{16\mu Q^2}{b\pi^2} \right)^{1/6} \quad (\text{independent of } L) \end{aligned}$$

(b) Obtain Murray's law by applying result from part (a) to each vessel at a bifurcation.

$$Q_1 = Q_2 + Q_3 \quad (\text{vessels labeled as in Fig. P2.10})$$

$$R_i \propto Q_i^{1/3} \text{ or } Q_i \propto R_i^3 \quad (\text{optimum found above})$$

$$R_1^3 = R_2^3 + R_3^3$$

(c) Extend Murray's law to generations j and k

Let Q_{ij} = flow rate in vessel i at generation j

Same total flow rate at each generation:

$$\sum_{i=1}^{N_j} Q_{ij} = \sum_{i=1}^{N_k} Q_{ik}$$

Solution: Chapter 2

Energy minimization:

$$Q_{ij} \propto R_{ij}^3$$

Combine:

$$\sum_{i=1}^{N_j} R_{ij}^3 = \sum_{i=1}^{N_k} R_{ik}^3$$

2.11 Open-Channel Flow

(a) Dimensional analysis

Governing quantities with dimensions:

$$\begin{array}{ll} [U] = \text{LT}^{-1} & [g] = \text{LT}^{-2} \\ [H] = \text{L} & [\rho] = \text{ML}^{-3} \\ [W] = \text{L} & [\mu] = \text{ML}^{-1}\text{T}^{-1} \\ [k] = \text{L} & [S] = \text{dimensionless (slope)} \end{array}$$

7 quantities – 3 dimensions = 4 dimensionless groups

For N_1 choose U (the unknown) for the numerator and g and H for the denominator:

$$N_1 = \frac{U}{g^a H^b}$$
$$[U] = \text{LT}^{-1} = [g^a H^b] = (\text{LT}^{-2})^a \text{L}^b = \text{L}^{a+b} \text{T}^{-2a}$$

$$\text{L: } 1 = a + b$$

$$\text{T: } -1 = -2a$$

$$a = 1/2 = b$$

$$N_1 = \frac{U}{(gH)^{1/2}} (= \text{Fr}^{1/2})$$

Note: The reason for preferring H over W or k is that U may be expected to be independent of W for $W/H \rightarrow \infty$ (as in a parallel-plate channel) and independent of k for $k/H \rightarrow 0$ (smooth surfaces), whereas H will always affect U . That g is a key quantity in a gravity-driven flow is obvious.

A known group with ρ and μ is Re:

$$N_2 = \frac{UH\rho}{\mu} = \text{Re}$$

The remaining two groups must include W and k . Let

$$N_3 = \frac{W}{H} = \phi \quad \text{aspect ratio}$$

$$N_4 = \frac{k}{H} \quad \text{roughness parameter}$$

Conclusion:

$$\frac{U}{(gH)^{1/2}} = F(S, \text{Re}, \phi, k/H)$$

Solution: Chapter 2

U will certainly increase with increasing slope and decrease with increasing roughness, but the form of F is largely unknown.

- (b) Force balance for length L of channel

Let τ_w be the shear stress averaged over the solid surfaces. Then

shear force = τ_w (side area + bottom area) = $\tau_w(2H + w)L$

gravitational force = (fluid mass) · (grav, accel. in flow direction)

$$= (\rho LHW)g \sin \theta = \rho LHW gS$$

$$\rho LHW gS = \tau_w(2H + W)L$$

$$\tau_w = \rho gS \left(\frac{HW}{2H + W} \right) = \frac{\rho gHS\phi}{2 + \phi}$$

- (c) Force balance using hydraulic diameter

Let $D_H = \frac{4A}{C}$ with A = cross-sectional area

C = wetted perimeter (solid only).

For the rectangular channel,

$$D_H = \frac{4HW}{2H + W} = \frac{4H\phi}{2 + \phi}$$

The general expression for τ_w is

$$\rho ALgS = \tau_w CL$$

$$\tau_w = \frac{\rho gSA}{C} = \frac{\rho gSD_H}{4}$$

- (d) Friction factor independent of flow rate (f independent of Re)

$$f = \frac{2\tau_w}{\rho U^2} = \frac{2}{\rho U^2} \left(\frac{\rho gSD_H}{4} \right) = \frac{gSD_H}{2U^2}$$

Arrange as in part (a), but with D_H instead of H :

$$\frac{U}{(gD_H)^{1/2}} = \left(\frac{S}{2f} \right)^{1/2}$$

It is seen now that $U \propto S^{1/2}$. Although independent of Re_H and presumably ϕ , in a rough channel f will depend on k/D_H [Eq. (P2.11-2)].