

***Introduction to the Micromechanics of Composite
Materials 1st edition Yin Solutions Manual***

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Chapter 2

Vectors and tensors

2.1 Exercise

1. Prove the following identity equations using the index notation:

$$\mathbf{a} \cdot (\mathbf{b} \mathbf{c}) = (\mathbf{a} \mathbf{b}) \cdot \mathbf{c}$$

$$\mathbf{a} (\mathbf{b} \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}$$

$$(\mathbf{a} \mathbf{b}) \cdot (\mathbf{c} \mathbf{d}) = \mathbf{a} \cdot [\mathbf{b} (\mathbf{c} \mathbf{d})]$$

Proof:

$$(1) \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_i (\varepsilon_{ijk} b_j c_k) = (\varepsilon_{ijk} a_i b_j) c_k = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

$$(2) \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \varepsilon_{ijk} a_j (\varepsilon_{klm} b_l c_m) = (\varepsilon_{ijk} \varepsilon_{lmk}) a_j b_l c_m = (\delta_{ij} \delta_{jm} - \delta_{im} \delta_{jl}) a_j b_l c_m \\ = a_j b_i c_j - a_j b_j c_i = (a_j c_j) b_i - (a_j b_j) c_i = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}$$

$$(3) (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\varepsilon_{ijk} a_i b_j) (\varepsilon_{klm} c_l d_m) = a_i (\varepsilon_{ijk} b_j (\varepsilon_{lmk} c_l d_m)) = \mathbf{a} \cdot [\mathbf{b} \times (\mathbf{c} \times \mathbf{d})]$$

2. For a rotational matrix $\mathbf{Q}$ between two Cartesian coordinates $\mathbf{x}$ and $\mathbf{x}'$ with a common origin, say $Q_{ij} = \cos(x_i, x'_j)$. The angular relationship between any two Cartesian reference frames $\mathbf{x}$ and $\mathbf{x}'$ can be represented in terms of three Euler angles (ϕ, θ, ψ) as follows: a) the first rotation is by an angle ϕ about the x_3 axis, b) the second rotation is by an angle $\theta \in (0, \pi)$ about the x_1 axis, and c) the third rotation is by an angle ψ about the x_3 axis again. Then the transformation matrix can be written as:

$$\mathbf{Q} = \begin{bmatrix} \cos \psi \cos \phi - \cos \theta \sin \psi \sin \phi & \cos \psi \sin \phi + \cos \theta \sin \psi \cos \phi & \sin \psi \sin \theta \\ -\sin \psi \cos \phi - \cos \theta \cos \psi \sin \phi & -\sin \psi \sin \phi + \cos \theta \cos \psi \cos \phi & \cos \psi \sin \theta \\ \sin \theta \sin \phi & -\sin \theta \cos \phi & \cos \theta \end{bmatrix}.$$

Prove

$$Q_{ik} Q_{jk} = \delta_{ij}.$$

Proof:

We calculate this one by one

$$Q_{1k}Q_{1k}$$

$$\begin{aligned}
&= (\cos \psi \cos \phi - \cos \theta \sin \psi \sin \phi)^2 + (\cos \psi \sin \phi + \cos \theta \sin \psi \cos \phi)^2 + (\sin \psi \sin \theta)^2 \\
&= \cos^2 \psi \cos^2 \phi - 2 \cos \psi \cos \phi \cos \theta \sin \psi \sin \phi + \cos^2 \theta \sin^2 \psi \cos^2 \phi \\
&\quad + \cos^2 \psi \sin^2 \phi + 2 \cos \theta \sin \psi \cos \phi + \cos^2 \theta \sin^2 \psi \cos^2 \phi + \sin^2 \psi \sin^2 \theta \\
&= (\cos^2 \psi \cos^2 \phi + \cos^2 \psi \sin^2 \phi) \\
&\quad + (\cos^2 \theta \sin^2 \psi \cos^2 \phi + \cos^2 \theta \sin^2 \psi \cos^2 \phi) + \sin^2 \psi \sin^2 \theta \\
&= \cos^2 \psi + \cos^2 \theta \sin^2 \psi + \sin^2 \psi \sin^2 \theta \\
&= 1
\end{aligned}$$

$$Q_{1k}Q_{2k}$$

$$\begin{aligned}
&= (\cos \psi \cos \phi - \cos \theta \sin \psi \sin \phi) (-\sin \psi \cos \phi - \cos \theta \cos \psi \sin \phi) \\
&\quad + (\cos \psi \sin \phi + \cos \theta \sin \psi \cos \phi) (-\sin \psi \sin \phi + \cos \theta \cos \psi \cos \phi) \\
&\quad + (\sin \psi \sin \theta) (\cos \psi \sin \theta) \\
&= -\cos \psi \cos^2 \phi \sin \psi - \cos^2 \psi \cos \phi \cos \theta \sin \phi + \cos \theta \sin^2 \psi \sin \phi \cos \phi + \cos^2 \theta \sin \psi \sin^2 \phi \cos \psi \\
&\quad - \cos \psi \sin^2 \phi \sin \psi + \cos^2 \psi \cos \phi \cos \theta \sin \phi - \cos \theta \sin^2 \psi \sin \phi \cos \phi + \cos^2 \theta \sin \psi \cos^2 \phi \cos \psi \\
&\quad + \sin \psi \cos \psi \sin^2 \theta \\
&= -(\cos \psi \cos^2 \phi \sin \psi + \cos \psi \sin^2 \phi \sin \psi) \\
&\quad + (\cos^2 \theta \sin \psi \sin^2 \phi \cos \psi + \cos^2 \theta \sin \psi \cos^2 \phi \cos \psi) \\
&\quad + \sin \psi \cos \psi \sin^2 \theta \\
&= -\cos \psi \sin \psi + \cos^2 \theta \sin \psi \cos \psi + \sin \psi \cos \psi \sin^2 \theta \\
&= 0
\end{aligned}$$

$$Q_{1k}Q_{3k}$$

$$\begin{aligned}
&= (\cos \psi \cos \phi - \cos \theta \sin \psi \sin \phi) (\sin \theta \sin \phi) \\
&\quad + (\cos \psi \sin \phi + \cos \theta \sin \psi \cos \phi) (-\sin \theta \cos \phi) \\
&\quad + (\sin \psi \sin \theta) \cos \theta \\
&= \cos \psi \cos \phi \sin \theta \sin \phi - \cos \theta \sin \psi \sin^2 \phi \sin \theta \\
&\quad - \cos \psi \sin \phi \sin \theta \cos \phi - \cos \theta \sin \psi \cos^2 \phi \sin \theta \\
&\quad + \sin \psi \sin \theta \cos \theta \\
&= -\cos \theta \sin \psi (\sin^2 \phi + \cos^2 \phi) \sin \theta + \sin \psi \sin \theta \cos \theta \\
&= 0
\end{aligned}$$

$$Q_{2k}Q_{2k}$$

$$= (-\sin \psi \cos \phi - \cos \theta \cos \psi \sin \phi) (-\sin \psi \sin \phi - \cos \theta \cos \psi \sin \phi)$$

$$\begin{aligned}
& +(\sin \psi \sin \phi + \cos \theta \cos \psi \cos \phi)(-\sin \psi \sin \phi + \cos \theta \cos \psi \cos \phi) \\
& +(\cos \psi \sin \theta)(\cos \psi \sin \theta) \\
& =\sin^2 \psi \cos^2 \phi + 2 \sin \psi \cos \phi \cos \theta \cos \psi \sin \phi + \cos^2 \theta \cos^2 \psi \sin^2 \phi \\
& +\sin^2 \psi \sin^2 \phi - 2 \sin \psi \sin \phi \cos \theta \cos \psi \cos \phi + \cos^2 \theta \cos^2 \psi \cos^2 \phi \\
& +\cos^2 \psi \sin^2 \theta \\
& =(\sin^2 \psi \cos^2 \phi + \sin^2 \psi \sin^2 \phi) \\
& +(\cos^2 \theta \cos^2 \psi \sin^2 \phi + \cos^2 \theta \cos^2 \psi \cos^2 \phi) \\
& +\cos^2 \psi \sin^2 \theta \\
& =\sin^2 \psi + \cos^2 \theta \cos^2 \psi + \cos^2 \psi \sin^2 \theta \\
& =1
\end{aligned}$$

$$\begin{aligned}
& Q_{2k}Q_{3k} \\
& =(-\sin \psi \cos \phi - \cos \theta \cos \psi \sin \phi)(\sin \theta \sin \phi) \\
& +(-\sin \psi \sin \phi + \cos \theta \cos \psi \cos \phi)(-\sin \theta \cos \phi) \\
& +(\cos \psi \sin \theta)(\cos \theta) \\
& =-\sin \psi \cos \phi \sin \theta \sin \phi - \cos \theta \cos \psi \sin \phi \sin \theta \sin \phi \\
& +\sin \psi \sin \phi \sin \theta \cos \phi - \cos \theta \cos \psi \cos \phi \sin \theta \cos \phi \\
& +\cos \psi \sin \theta \cos \theta \\
& =-\cos \theta \cos \psi (\sin^2 \phi + \cos^2 \phi) \sin \theta + \cos \psi \sin \theta \cos \theta \\
& =0
\end{aligned}$$

$$\begin{aligned}
& Q_{3k}Q_{3k} \\
& =(\sin \theta \sin \phi)^2 + (-\sin \theta \cos \phi)^2 + \cos^2 \theta \\
& =\sin^2 \theta (\sin^2 \phi + \cos^2 \phi) + \cos^2 \theta \\
& =1
\end{aligned}$$

And for symmetry

$$Q_{2k}Q_{1k}=Q_{1k}Q_{2k}=0$$

$$Q_{3k}Q_{1k}=Q_{1k}Q_{3k}=0$$

$$Q_{3k}Q_{2k}=Q_{2k}Q_{3k}=0$$

So

$$Q_{ik}Q_{jk}=\delta_{ij}$$

3. Prove the following tensors are isotropic for the corresponding tensor ranks:

$$A_{ij} = k\delta_{ij}; \quad A_{ijk} = ke_{ijk}; \quad A_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk};$$

Proof:

1. We need to point out that the transformation matrix $\mathbf{Q}$, it satisfies

$$Q_{mi}Q_{ni} = \delta_{mn}$$

For $\lambda\delta_{ij}$, under the transformation matrix $\mathbf{Q}$, it becomes

$$Q_{mi}Q_{nj}\lambda\delta_{ij} = \lambda Q_{mi}Q_{ni} = \lambda\delta_{mn} = \lambda\delta_{ij}$$

Since $\mathbf{Q}$ is transformation matrix, it satisfies $Q_{mi}Q_{ni} = \delta_{mn}$

2. For $A_{ijk} = \lambda e_{ijk}$ under a transformation, we can write

$$A'_{lmn} = \lambda Q_{li}Q_{mj}Q_{nk}e_{ijk}$$

Considering $e_{ijk} = (\mathbf{e}_i \times \mathbf{e}_j) \cdot \mathbf{e}_k$, we can write

$$\begin{aligned} A'_{lmn} &= \lambda Q_{li}Q_{mj}Q_{nk} (\mathbf{e}_i \times \mathbf{e}_j) \cdot \mathbf{e}_k \\ &= \lambda (Q_{li}\mathbf{e}_i \times Q_{mj}\mathbf{e}_j) \cdot Q_{nk}\mathbf{e}_k \\ &= \lambda (\mathbf{e}'_l \times \mathbf{e}'_m) \cdot \mathbf{e}'_n \end{aligned}$$

Considering the properties of the base vector, $(\mathbf{e}'_l \times \mathbf{e}'_m) \cdot \mathbf{e}'_n = (\mathbf{e}_l \times \mathbf{e}_m) \cdot \mathbf{e}_n$ as the volume of the cubes of the vectors, we can write

$$A'_{lmn} = \lambda (\mathbf{e}'_l \times \mathbf{e}'_m) \cdot \mathbf{e}'_n = \lambda (\mathbf{e}_l \times \mathbf{e}_m) \cdot \mathbf{e}_n = A_{lmn}$$

So it is isotropic

3. For fourth rank tensor $A_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$

$$\begin{aligned} A'_{mnst} &= Q_{mi}Q_{ni}Q_{sk}Q_{tk} (\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}) \\ &= \alpha Q_{mi}Q_{ni}Q_{sk}Q_{tk} + \beta Q_{mi}Q_{nj}Q_{si}Q_{tj} + \gamma Q_{mi}Q_{nj}Q_{sj}Q_{ti} \\ &= \alpha (Q_{mi}Q_{ni}) (Q_{sk}Q_{tk}) + \beta (Q_{mi}Q_{si}) (Q_{nj}Q_{tj}) + \gamma (Q_{mi}Q_{ti}) (Q_{bj}Q_{sj}) \\ &= \alpha\delta_{mn}\delta_{st} + \beta\delta_{ms}\delta_{nt} + \gamma\delta_{mt}\delta_{ns} \\ &= A_{mnst} \end{aligned}$$

So it is isotropic

4. Consider vectors $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (3, 2, 1)$. Determine the following quantities:

$$\mathbf{a} \cdot \mathbf{b}, \mathbf{a} \times \mathbf{b}, \mathbf{a} \otimes \mathbf{b}.$$

Now if the coordinate is rotated along the axis x_3 at 30° counterclockwise, i.e. the $x_1 - x_2$ plane is rotated 30° counterclockwise and the x_3 axis remains the same, recalculate the above quantities in the new coordinate system.

Answer:

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i = 1 \times 3 + 2 \times 2 + 3 \times 1 = 10$$

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \varepsilon_{kij} a_i b_j = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} = (2 \times 1 - 3 \times 2) i + (3 \times 3 - 1 \times 1) j + (1 \times 2 - 2 \times 3) k = \\ &= (-4, 8, -4) \end{aligned}$$

$$\mathbf{a} \otimes \mathbf{b} = a_i b_j = \begin{bmatrix} 1 \times 3 & 1 \times 2 & 1 \times 1 \\ 2 \times 3 & 2 \times 2 & 2 \times 1 \\ 3 \times 3 & 3 \times 2 & 3 \times 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \\ 9 & 6 & 3 \end{bmatrix}$$

Then we consider the condition of rotation,

The rotation matrix is

$$Q = \begin{bmatrix} \cos a & -\sin a & \\ \sin a & \cos a & \\ & & 1 \end{bmatrix} = \begin{bmatrix} \cos 30 & -\sin 30 & \\ \sin 30 & \cos 30 & \\ & & 1 \end{bmatrix}$$

After the rotation of the coordinate, the new quantities of the vectors would be

$$\mathbf{a} = (1, 2, 3) \begin{bmatrix} \cos 30 & -\sin 30 & \\ \sin 30 & \cos 30 & \\ & & 1 \end{bmatrix} = \left(\frac{\sqrt{3}}{2} + 1, -\frac{1}{2} + \sqrt{3}, 3 \right) = (1.8660 \ 1.2321 \ 3.0000)$$

$$\mathbf{b} = (3, 2, 1) \begin{bmatrix} \cos 30 & -\sin 30 & \\ \sin 30 & \cos 30 & \\ & & 1 \end{bmatrix} = \left(\frac{3\sqrt{3}}{2} + 1, -\frac{3}{2} + \sqrt{3}, 1 \right) = (3.5981 \ 0.2321 \ 1.0000)$$

Then

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= a_i b_i = \left(\frac{\sqrt{3}}{2} + 1 \right) \times \left(\frac{3\sqrt{3}}{2} + 1 \right) + \left(-\frac{1}{2} + \sqrt{3} \right) \times \left(-\frac{3}{2} + \sqrt{3} \right) + 3 \times 1 \\ &= \left(\frac{13}{4} + 2\sqrt{3} \right) + \left(\frac{15}{4} - 2\sqrt{3} \right) + 3 = 10 \end{aligned}$$

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \varepsilon_{kij} a_i b_j = \begin{vmatrix} i & j & k \\ \frac{\sqrt{3}}{2} + 1 & -\frac{1}{2} + \sqrt{3} & 3 \\ \frac{3\sqrt{3}}{2} + 1 & -\frac{3}{2} + \sqrt{3} & 1 \end{vmatrix} \\ &= \left[-\frac{1}{2} + \sqrt{3} - \left(-\frac{9}{2} + 3\sqrt{3} \right) \right] i + \left[\frac{9\sqrt{3}}{2} + 3 - \left(\frac{\sqrt{3}}{2} + 1 \right) \right] j + \left[\frac{\sqrt{3}}{4} - \left(4 + \frac{\sqrt{3}}{4} \right) \right] k \\ &= (4 - 2\sqrt{3}, 4\sqrt{3} + 2, -4) = (0.5359 \ 8.9282 \ -4.0000) \end{aligned}$$

Note that this quantity of the vector is in the new coordinate system

$$\begin{aligned} \mathbf{a} \otimes \mathbf{b} &= a_i b_j = \begin{bmatrix} \left(\frac{\sqrt{3}}{2} + 1 \right) \times \left(\frac{3\sqrt{3}}{2} + 1 \right) & \left(\frac{3\sqrt{3}}{2} + 1 \right) \times \left(-\frac{3}{2} + \sqrt{3} \right) & \left(\frac{\sqrt{3}}{2} + 1 \right) \times 1 \\ \left(-\frac{1}{2} + \sqrt{3} \right) \times \left(\frac{3\sqrt{3}}{2} + 1 \right) & \left(-\frac{1}{2} + \sqrt{3} \right) \times \left(-\frac{3}{2} + \sqrt{3} \right) & \left(-\frac{1}{2} + \sqrt{3} \right) \times 1 \\ 3 \times \left(\frac{3\sqrt{3}}{2} + 1 \right) & 3 \times \left(-\frac{3}{2} + \sqrt{3} \right) & 3 \times 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{13}{4} + 2\sqrt{3} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{2} + 1 \\ 4 + \frac{\sqrt{3}}{4} & \frac{15}{4} - 2\sqrt{3} & -\frac{1}{2} + \sqrt{3} \\ \frac{9\sqrt{3}}{2} + 3 & -\frac{9}{2} + 3\sqrt{3} & 3 \end{bmatrix} = \begin{bmatrix} 6.7141 & 0.4330 & 1.8660 \\ 4.4330 & 0.2859 & 1.2321 \\ 10.7942 & 0.6962 & 3.0000 \end{bmatrix} \end{aligned}$$

Still in the new coordinate system.

5. An orthogonal material has the stiffness tensor given in form of $C_{ijkl} = C_{IK}^1 \delta_{ij} \delta_{kl} + C_{IJ}^2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$ with

$$\mathbf{C}^1 = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1.5 & 0.75 \\ 0.5 & 0.75 & 2 \end{bmatrix}; \quad \mathbf{C}^2 = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.75 & 0.5 \\ 0.25 & 0.5 & 1 \end{bmatrix}.$$

- (a) Determine the components of the stiffness in form of a 6 by 6 matrix.
- (b) Calculate the components of the compliance.
- (c) Write the compliance tensor in the form of $D_{ijkl} = D_{IK}^1 \delta_{ij} \delta_{kl} + D_{IJ}^2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$. Determine D_{IK}^1 and D_{IJ}^2 in terms of C_{IK}^1 and C_{IJ}^2 . Check whether the calculation in (b) is consistent with the formulation.

Answer:

(a) The stiffness matrix transfers a strain tensor to a stress tensor as $\sigma_{ij} = C_{ijkl} \epsilon_{kl}$. Due to the symmetry of stress and strain, we can use a list of 6 components to represent the 3x3 tensor to save space in numerical programming as follows:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3331} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2331} & C_{2312} \\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3131} & C_{3112} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1231} & C_{1212} \end{bmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \\ 2\epsilon_{12} \end{pmatrix}$$

Notice that the strain list is the engineering strain instead of the tensorial strain, whose difference is in the shear strain components, i.e. $\gamma_{23} = 2\epsilon_{23}$, etc. but the normal strain ϵ keep the same. If tensorial strain is used, the above relation becomes

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 2C_{1123} & 2C_{1131} & 2C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & 2C_{2223} & 2C_{2231} & 2C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & 2C_{3323} & 2C_{3331} & 2C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & 2C_{2323} & 2C_{2331} & 2C_{2312} \\ C_{3111} & C_{3122} & C_{3133} & 2C_{3123} & 2C_{3131} & 2C_{3112} \\ C_{1211} & C_{1222} & C_{1233} & 2C_{1223} & 2C_{1231} & 2C_{1212} \end{bmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{31} \\ \epsilon_{12} \end{pmatrix}$$

where the symmetry of the stiffness tensor is lost. Therefore, we generally use the engineering strain in the 6-component notation. However, for orthotropic materials, because the components without symmetry are zero, this issue is not real. In the following, let us use engineering strain

$$C_{ijkl} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3331} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2331} & C_{2312} \\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3131} & C_{3112} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1231} & C_{1212} \end{bmatrix}$$

$$= \begin{bmatrix} C_{11}^1 + C_{11}^2 \times 2 & C_{12}^1 & C_{13}^1 & 0 & 0 & 0 \\ C_{21}^1 & C_{22}^1 + C_{22}^2 \times 2 & C_{32}^1 & 0 & 0 & 0 \\ C_{31}^1 & C_{32}^1 & C_{33}^1 + C_{33}^2 \times 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{23}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{13}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{12}^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 0.5 \times 2 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 1.5 + 0.75 \times 2 & 0.75 & 0 & 0 & 0 \\ 0.5 & 0.75 & 1 + 0.5 \times 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 3 & 0.75 & 0 & 0 & 0 \\ 0.5 & 0.75 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.25 \end{bmatrix}$$

(b)

Using Matlab, we calculate the transverse if the stiffness matrix, compliance matrix D

$$D = C^{-1} = \begin{bmatrix} 0.5320 & -0.0756 & -0.0523 & 0 & 0 & 0 \\ -0.0756 & 0.3605 & -0.0581 & 0 & 0 & 0 \\ -0.0523 & -0.0581 & 0.2674 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Notice that this is for engineering strain. If tensorial strain is used, the bottom-right quadrant shall be applied a multiplier of 1/2.

(c)

There are not definite relationship between D_{IK}^1 , D_{IJ}^2 and C_{IK}^1 , C_{IJ}^2

From the calculation above, we see that the relationship between C_{IK}^1 , C_{IJ}^2 and C_{ijkl} is not one to

one.

There are numerous sets of C_{IK}^1 and C_{IJ}^2 that can get the same C_{ijkl} as long as $C_{ii}^1 + 2C_{ii}^2$ ($i = 1, 2, 3$)

keeps the same value.

The same is for D_{IK}^1 and D_{IJ}^2 , their composition can be various and may not be determined uniquely.

However, we can still get the one way to obtain the general relation between C_{IK}^1 , C_{IJ}^2 and D_{IK}^1 , D_{IJ}^2

Let

$$C^3 = \begin{bmatrix} C_{11}^1 + C_{11}^2 \times 2 & C_{12}^1 & C_{13}^1 \\ C_{21}^1 & C_{22}^1 + C_{22}^2 \times 2 & C_{32}^1 \\ C_{31}^1 & C_{32}^1 & C_{33}^1 + C_{33}^2 \times 2 \end{bmatrix}$$

$$D^3 = \begin{bmatrix} D_{11}^1 + D_{11}^2 \times 2 & D_{12}^1 & D_{13}^1 \\ D_{21}^1 & D_{22}^1 + D_{22}^2 \times 2 & D_{32}^1 \\ D_{31}^1 & D_{32}^1 & D_{33}^1 + D_{33}^2 \times 2 \end{bmatrix}$$

$$C^4 = \begin{bmatrix} C_{23}^2 & & \\ & C_{13}^2 & \\ & & C_{12}^2 \end{bmatrix}$$

$$D^4 = \begin{bmatrix} D_{23}^2 & & \\ & D_{13}^2 & \\ & & D_{12}^2 \end{bmatrix}$$

Since $CD = I$

We have

$$C^3 D^3 = I$$

$$C^4 D^4 = I$$

So when $C_{IK}^1 C_{IJ}^2$ is determined, we can calculate get $D^3 D^4$, based on D^4 we get

$$\begin{cases} D_{23}^2 = D_{32}^2 = (C_{23}^2)^{-1} \\ D_{13}^2 = D_{31}^2 = (C_{13}^2)^{-1} \\ D_{12}^2 = D_{21}^2 = (C_{12}^2)^{-1} \end{cases}$$

Then $D_{11}^2 D_{22}^2 D_{33}^2$, could be chosen to be any value

From D^3 we get

$$D_{ij}^1 = D_{ij}^3 \quad (i \neq j)$$

And

$$\begin{cases} D_{11}^1 = D_{11}^3 - 2 \times D_{11}^2 \\ D_{22}^1 = D_{22}^3 - 2 \times D_{22}^2 \\ D_{33}^1 = D_{33}^3 - 2 \times D_{33}^2 \end{cases}$$

So by now we solve the problem. If we have the value of $C_{IK}^1 C_{IJ}^2$, we can get the according value for

$D_{IK}^1 D_{IJ}^2$, but they are not uniquely determined. From the above derivation, we can also provide an analytical form for the case of tensorial strains.

6. Prove the following identities

$$\nabla \times (\nabla \phi) = \mathbf{0},$$

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0,$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla (\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$$

where ϕ is a scalar field and $\mathbf{a}$ a vector field.

Proof:

$$(1) \nabla \times (\nabla \phi) = \varepsilon_{ijk} \phi_{,kj} = -\varepsilon_{ikj} \phi_{,kj} = -\varepsilon_{ijk} \phi_{,kj} = 0 \text{ (Switch } j, k \text{ in the last two formulas)}$$

$$(2) \nabla \cdot (\nabla \times \mathbf{a}) = (\varepsilon_{ijk} a_{j,k})_{,i} = \varepsilon_{ijk} a_{j,ki} = -\varepsilon_{kji} a_{j,ik} = -\varepsilon_{ijk} a_{j,ki} = 0 \text{ (Switch } i, k \text{ in the last two formulas)}$$

$$\begin{aligned}
(3) \nabla \times (\nabla \times \mathbf{a}) &= \varepsilon_{ijk}(\varepsilon_{kmn}a_{n,m})_{,j} = (\varepsilon_{ijk}\varepsilon_{kmn})a_{n,mj} \\
&= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})a_{n,mj} = a_{j,ij} - a_{i,jj} = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}
\end{aligned}$$

7. Given a scalar field in a 3-dimensional infinite domain as $\psi = |\mathbf{x} - \mathbf{x}^0|$, which is a distance from $\mathbf{x}$ to a known point $\mathbf{x}^0$, derive $\nabla \nabla \psi$, which is a second rank tensor. Similarly, for $\phi = 1/|\mathbf{x} - \mathbf{x}^0|$, derive $\nabla \nabla \phi$.

Answer:

$$\text{Let } r = |\mathbf{x} - \mathbf{x}^0|^2$$

First we get

$$\begin{aligned}
\nabla \psi &= \frac{d\psi}{dr} \frac{dr}{dx_i} = \frac{1}{2|\mathbf{x} - \mathbf{x}^0|} \cdot 2(x_i - x_i^0) \\
&= \frac{(x_i - x_i^0)}{|\mathbf{x} - \mathbf{x}^0|} \\
\nabla \nabla \psi &= -\frac{1}{2|\mathbf{x} - \mathbf{x}^0|^3} (x_i - x_i^0) \cdot 2(x_j - x_j^0) + \frac{\delta_{ij}}{|\mathbf{x} - \mathbf{x}^0|} \\
&= \frac{-(x_i - x_i^0)(x_j - x_j^0) + \delta_{ij}|\mathbf{x} - \mathbf{x}^0|^2}{|\mathbf{x} - \mathbf{x}^0|^3}
\end{aligned}$$

$$\text{For } \phi = \frac{1}{|\mathbf{x} - \mathbf{x}^0|}$$

$$\begin{aligned}
\nabla \psi &= \frac{d\psi}{dr} \frac{dr}{dx_i} = -\frac{1}{2|\mathbf{x} - \mathbf{x}^0|} \cdot 2(x_i - x_i^0) \\
&= \frac{-(x_i - x_i^0)}{|\mathbf{x} - \mathbf{x}^0|^3} \\
\nabla \nabla \psi &= \frac{3}{2|\mathbf{x} - \mathbf{x}^0|^5} (x_i - x_i^0) \cdot 2(x_j - x_j^0) + \frac{\delta_{ij}}{|\mathbf{x} - \mathbf{x}^0|^3} \\
&= \frac{3(x_i - x_i^0)(x_j - x_j^0) - \delta_{ij}|\mathbf{x} - \mathbf{x}^0|^2}{|\mathbf{x} - \mathbf{x}^0|^5}
\end{aligned}$$

8. Given a vector field $\mathbf{a} = \mathbf{x}$ in a spherical domain Ω : $x_1^2 + x_2^2 + x_3^2 \leq 1$, use the Helmholtz's decomposition to write $\mathbf{a} = \nabla \varphi + \nabla \times \mathbf{b}$ in terms of a scalar potential and a vector potential $\mathbf{b}$. Find the specific form of φ and $\mathbf{b}$ in terms of x_i .

Answer:

From the textbook, we have

$$a(x) = \nabla^2 u(x) = \nabla(\nabla \cdot u(x)) - \nabla \times (\nabla \times u(x))$$

We choose

$$\varphi = \nabla \cdot u(x)$$

$$b = -\nabla \times u(x)$$

Then we can get

$$a(x) = \nabla^2 u(x) = \nabla \varphi + \nabla \times b$$

So now we calculate $u(x)$

$$\nabla^2 u(x) = a(x) = x$$

We can get

$$u(x) = \left(\frac{x_1^3}{6} + x_1 f_1(x_2, x_3) + g_1(x_2, x_3), \frac{x_2^3}{6} + x_2 f_2(x_1, x_3) + g_2(x_1, x_3), \frac{x_3^3}{6} + x_3 f_3(x_1, x_2) + g_3(x_1, x_2) \right)$$

Here $f_1(x_2, x_3)$ and $g_1(x_2, x_3)$ are any functions

$$\varphi = \nabla \cdot u(x) = \frac{x_1^2}{2} + f_1(x_2, x_3) + \frac{x_2^2}{2} + f_2(x_1, x_3) + \frac{x_3^2}{2} + f_3(x_1, x_2)$$

$$b = -\nabla \times u(\mathbf{x}) = - \begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1(\mathbf{x}) & u_2(\mathbf{x}) & u_3(\mathbf{x}) \\ i & j & k \end{vmatrix}$$

$$= \begin{bmatrix} - \left(x_3 \frac{\partial f_3(x_1, x_2)}{\partial x_2} + \frac{\partial g_3(x_1, x_2)}{\partial x_2} - x_2 \frac{\partial f_2(x_1, x_3)}{\partial x_3} - \frac{\partial g_2(x_1, x_3)}{\partial x_3} \right) \\ x_2 \frac{\partial f_1(x_2, x_3)}{\partial x_3} + \frac{\partial g_1(x_2, x_3)}{\partial x_3} - x_3 \frac{\partial f_3(x_1, x_2)}{\partial x_1} - \frac{\partial g_3(x_1, x_2)}{\partial x_1} \\ x_2 \frac{\partial f_2(x_1, x_3)}{\partial x_1} + \frac{\partial g_2(x_1, x_3)}{\partial x_1} - x_1 \frac{\partial f_1(x_2, x_3)}{\partial x_2} - \frac{\partial g_1(x_2, x_3)}{\partial x_2} \end{bmatrix}$$

9. Consider a scalar field $\varphi = x_1^2 + x_2^2 + x_3^2$ in a spherical domain : $x_1^2 + x_2^2 + x_3^2 \leq 1$. Confirm the following equations:

$$\int_{\Omega} \nabla \varphi dV = \int_{\partial \Omega} \mathbf{n} \varphi dS$$

$$\int_{\Omega} \nabla^2 \varphi dV = \int_{\partial \Omega} \mathbf{n} \cdot \nabla \varphi dS$$

Answer:

(1)

For the left hand side

$$\int \nabla \varphi dV = \int 2x_i dV$$

$$= \int_{r=0}^1 \int_{\theta=0}^{\pi} \int_{\alpha=0}^{2\pi} (2r \sin \theta \cos \alpha) r^2 \sin \theta dr d\theta d\alpha \mathbf{i}$$

$$+ \int_{r=0}^1 \int_{\theta=0}^{\pi} \int_{\alpha=0}^{2\pi} (2r \sin \theta \sin \alpha) r^2 \sin \theta dr d\theta d\alpha \mathbf{j}$$

$$+ \int_{r=0}^1 \int_{\theta=0}^{\pi} \int_{\alpha=0}^{2\pi} (2r \cos \theta) r^2 \sin \theta dr d\theta d\alpha \mathbf{k}$$

$$= 0 + 0 + 0 = 0$$

For the right hand side

$$\int_{\partial} \mathbf{n} \varphi dV$$

$$= \int_{\theta=0}^{\pi} \int_{\alpha=0}^{2\pi} (\sin \theta \cos \alpha) \sin \theta d\theta d\alpha \mathbf{i}$$

$$+ \int_{\theta=0}^{\pi} \int_{\alpha=0}^{2\pi} (\sin \theta \sin \alpha) \sin \theta d\theta d\alpha \mathbf{j}$$

$$+ \int_{\theta=0}^{\pi} \int_{\alpha=0}^{2\pi} (\sin \theta \cos \alpha) r^2 \sin \theta d\theta d\alpha \mathbf{k}$$

$$=0+0+0=0$$

They are equal

(2)

For the left hand side

$$\int_{\Omega} \nabla^2 \varphi dV = \int_{\Omega} 6 dV = 8\pi$$

while

$$\int_{\partial\Omega} \mathbf{n} \cdot \nabla \varphi dS = \int_{\theta=0}^{\theta=\pi} \int_{\alpha=0}^{\alpha=2\pi} \sin \theta d\theta d\alpha = 2\pi \int_0^\pi \sin \theta d\theta = 2\pi$$

$$= \int_{\theta=0}^{\theta=\pi} \int_{\alpha=0}^{\alpha=2\pi} \sin \theta d\theta d\alpha = 2\pi \int_0^\pi \sin \theta d\theta = 2\pi$$

$$= 8\pi$$

They are equal.

10. Prove $G(\mathbf{x}, \mathbf{x}') = \ln \frac{1}{|\mathbf{x} - \mathbf{x}'|}$ is the Green's function for the 2-dimensional Laplace equation.

Proof:

$$G(\mathbf{x}, \mathbf{x}') = \ln \frac{1}{|\mathbf{x} - \mathbf{x}'|}$$

When $\mathbf{x} \neq \mathbf{x}'$

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') = \frac{\partial}{\partial x_i} \left\{ |x - x'| \frac{\partial}{\partial x_i} [(x - x')^2]^{-\frac{1}{2}} \right\}$$

$$= \frac{\partial}{\partial x_i} \left\{ |x - x'| \left(-\frac{1}{2}\right) [(x - x')^2]^{-\frac{3}{2}} 2(x_i - x'_i) \right\}$$

$$= -\frac{\partial}{\partial x_i} \frac{(x_i - x'_i)^2}{|x - x'|^2}$$

$$= -\frac{2}{|x - x'|^2} - \left\{ -1 [(x - x')^2]^{-2} 2(x_i - x'_i)^2 \right\}$$

$$= -\frac{2}{|x - x'|^2} + \frac{2(x_i - x'_i)^2}{|x - x'|^4}$$

$$= -\frac{2}{|x - x'|^2} + \frac{2|x - x'|^2}{|x - x'|^4}$$

$$= 0$$

Here since it's two dimensional, $\frac{\partial}{\partial x_i} x_i = 2$

When $\mathbf{x} = \mathbf{x}'$, this point is nondifferentiable

From above we know that

$$\frac{\partial G(x, x')}{\partial x_i} = -\frac{(x_i - x'_i)}{|x - x'|^2}$$

If we take derivative with $\mathbf{x}'$, then it is

$$\frac{\partial G(x, x')}{\partial x'_i} = -\frac{(x_i - x'_i)}{|x - x'|^2}$$

Using Gauss equation

$$\int_{\Omega} \nabla^2 G(x, x') dx' = \int_{\partial\Omega} \nabla G(x, x') \mathbf{n} dx' = \int_{\partial\Omega} \frac{(x_i - x'_i)}{|x - x'|^2} n_i dx'$$

Since $\nabla^2 G(x, x') = 0$ everywhere except $x=x'$, we can choose any $\partial\Omega$. Just choose it to be a circle

radius of one and centered at x ,

Then

$$\begin{aligned} & \int_{\Omega} \nabla^2 G(x, x') dx' \\ &= \int_{\partial\Omega} \frac{(x_i - x'_i)}{|x - x'|^2} n_i dx' \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=2\pi} -(\sin^2 \theta \cos^2 \varphi + \sin^2 \theta \sin^2 \varphi + \cos^2 \theta) \sin \theta d\theta d\varphi \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=2\pi} -1 \sin \theta d\theta d\varphi \\ &= -2\pi \end{aligned}$$

So it is Green function in two dimension

And it is better written in the form of

$$G(\mathbf{x}, \mathbf{x}') = -\frac{1}{2\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{x}'|}$$

11. Derive the Green's function for a cantilever beam bending.

Answer:

For the cantilever, using the same procedure in class, we have

$$EI \cdot G(x - x') = \frac{(x - x')^3}{6} H(x - x') + \frac{C_1}{6} x^3 + \frac{C_2}{2} x^2 + C_3 x + C_4$$

There is no displacement and tilt angle at the starting point $x=0$,

And the moment caused by this point force is x' at $x=0$ and 0 at $x=L$

So boundary condition is

$$\begin{cases} G(x - x') = 0 (x = 0) \\ G'(x - x') = 0 (x = 0) \\ G''(x - x') = x' (x = 0) \\ G'''(x - x') = 0 (x = L) \end{cases}$$

From these we can determine

$$\begin{cases} C_1 = -1 \\ C_2 = x' \\ C_3 = 0 \\ C_4 = 0 \end{cases}$$

So the Green function is

$$G(x - x') = \frac{1}{EI} \left[\frac{(x - x')^3}{6} H(x - x') + \frac{1}{6} x^3 + \frac{x'}{2} x^2 \right]$$

The beam's deflection shape function can be written as

$$w = \int_{\Omega} G(x, x') q(x') dx'$$

12. Write the 6 equations implied by compatibility condition for strains $\nabla \times \epsilon \times \nabla = \mathbf{0}$.

Answer:

$$\nabla \times \epsilon \times \nabla = \left(\epsilon_{kim} \frac{\partial \epsilon_{ij}}{\partial x_k} \right) \epsilon_{jln} \frac{\partial}{\partial x_l} = (\epsilon_{kim} \epsilon_{jnl}) \partial \epsilon_{ij,kl} = \mathbf{0}$$

when $(m = 1, n = 1)$

$$(\epsilon_{ki1} \epsilon_{j1l}) \partial \epsilon_{ij,kl} = (\epsilon_{231} \epsilon_{213}) \partial \epsilon_{32,23} (k = 2, i = 3, j = 2, l = 3)$$

$$+ (\epsilon_{231} \epsilon_{312}) \partial \epsilon_{33,22} (k = 2, i = 3, j = 3, l = 2)$$

$$+ (\epsilon_{321} \epsilon_{213}) \partial \epsilon_{22,33} (k = 3, i = 2, j = 2, l = 3)$$

$$+ (\epsilon_{321} \epsilon_{312}) \partial \epsilon_{23,32} (k = 3, i = 2, j = 3, l = 2)$$

$$= -\epsilon_{32,23} + \epsilon_{33,22} + \epsilon_{22,33} - \epsilon_{23,32}$$

$$= 0$$

So similar when $(m = 2, n = 2)$

We get

$$-\epsilon_{21,12} + \epsilon_{22,11} + \epsilon_{11,22} - \epsilon_{12,21} = 0$$

When $(m = 3, n = 3)$

$$-\epsilon_{13,31} + \epsilon_{11,33} + \epsilon_{33,11} - \epsilon_{31,13} = 0$$

$(m = 1, n = 2)$

$$(\epsilon_{ki1} \epsilon_{j2l}) \partial \epsilon_{ij,kl} = (\epsilon_{231} \epsilon_{213}) \partial \epsilon_{31,23} (k = 2, i = 3, j = 1, l = 3)$$

$$+ (\epsilon_{231} \epsilon_{321}) \partial \epsilon_{33,21} (k = 2, i = 3, j = 3, l = 1)$$

$$+ (\epsilon_{321} \epsilon_{123}) \partial \epsilon_{21,33} (k = 3, i = 2, j = 1, l = 3)$$

$$+ (\epsilon_{321} \epsilon_{321}) \partial \epsilon_{23,31} (k = 3, i = 2, j = 3, l = 1)$$

When $(m = 3, n = 1)$

$$\epsilon_{23,12} + \epsilon_{22,33} + \epsilon_{13,22} - \epsilon_{12,23} = 0$$

In conclude the six equations we get are:

$$\begin{cases} \epsilon_{33,22} + \epsilon_{22,33} - 2\epsilon_{23,32} = 0 \\ \epsilon_{22,11} + \epsilon_{11,22} - 2\epsilon_{12,21} = 0 \\ \epsilon_{11,33} + \epsilon_{33,11} - 2\epsilon_{31,13} = 0 \\ \epsilon_{31,23} - \epsilon_{33,21} - \epsilon_{21,33} + \epsilon_{23,31} = 0 \\ \epsilon_{12,31} - \epsilon_{11,32} - \epsilon_{32,11} + \epsilon_{23,12} = 0 \\ \epsilon_{12,31} - \epsilon_{11,32} - \epsilon_{32,11} + \epsilon_{31,12} = 0 \end{cases}$$

13. Solve the boundary value problem: For a solid elastic ball fully boned by an outer layer, the radius of the ball and outer surface radii are written as $R_i < R_o$, respectively. The

Young's moduli and Poisson's ratios of the ball and outer layer are respectively written as E_i and ν_i and E_o and ν_o . When a pressure P is applied on the outer surface, derive the displacement distribution along the radial direction in two materials.

Answer:

For strain condition

$$\varepsilon_{rr} = u_{r,r}$$

$$\varepsilon_{\theta\theta} = \varepsilon_{\varphi\varphi} = \frac{u_r}{r}$$

Equilibrium equation

$$\sigma_{rr,r} + \frac{2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi}}{r} = 0$$

Constitutive equation

$$\sigma_{rr} = \lambda (\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{\varphi\varphi}) + 2\mu \varepsilon_{rr} = (2\mu + \lambda) u_{r,r} + 2\lambda \frac{u_r}{r}$$

Here to be convenient, we use λ μ to replace E ν

The relationship is

$$\lambda = \frac{Ev}{(1+\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)}$$

According to the textbook, the displacement of this kind of problem could be written as

$$u_r = C_1 r + \frac{C_2}{r^2} \quad \text{Out layer}$$

$$u_r^* = C_1^* r + \frac{C_2^*}{r^2} \quad \text{inside sphere}$$

Assure the stress between the two different materials is $\sigma_{rr}|_{r=Ri} = P^*$

We have two separate problems

$$u_r = C_1 r + \frac{C_2}{r^2} \quad Ri \leq r \leq Ro$$

$$\begin{cases} \sigma_{rr}|_{r=Ro} = -P \\ \sigma_{rr}|_{r=Ri} = -P^* \end{cases} \quad (1)$$

$$u_r^* = C_1^* r + \frac{C_2^*}{r^2} \quad r \leq Ri$$

$$\begin{cases} u_r^*|_{r=0} = 0 \\ \sigma_{rr}^*|_{r=Ri} = P^* \end{cases} \quad (2)$$

$$u_r = u_r^*|_{r=Ri}$$

For the first boundary value problem, we get

$$C_1 = \frac{R_o^3 P^* - (2R_o^3 - R_i^3)P}{(2\mu_o + 3\lambda_o)(R_o^3 - R_i^3)}$$

$$C_2 = \frac{R_o^3 R_i^3 (P^* - P)}{4\mu_o (R_o^3 - R_i^3)}$$

$$u_r = \frac{R_i^3 P^* - (2R_o^3 - R_i^3)P}{(2\mu_o + 3\lambda_o)(R_o^3 - R_i^3)} r + \frac{R_o^3 R_i^3 (P^* - P)}{4\mu_o (R_o^3 - R_i^3)} \frac{1}{r^2}$$

For the second boundary condition,

At $r=0$ there is no displacement, so we can get $C_2^* = \frac{P^*}{2\mu_i + 3\lambda_i}$

$$u_r^* = \frac{P^*}{2\mu + 3\lambda} r$$

Then the problem is to decide P^*

The displacement of the two material at the interface must be equal

$$u_r = u_r^*|_{r=R_i}$$

So we have

$$\frac{R_i^3 P^* - (2R_o^3 - R_i^3)P}{(2\mu_o + 3\lambda_o)(R_o^3 - R_i^3)} R_i + \frac{R_o^3 R_i^3 (P^* - P)}{4\mu_o (R_o^3 - R_i^3)} \frac{1}{R_i^2} = \frac{P^*}{2\mu + 3\lambda} R_i$$

From this we get

$$P^* = \left[\left(\frac{(2R_o^3 - R_i^3)R_i}{(2\mu_o + 3\lambda_o)(R_o^3 - R_i^3)} + \frac{R_o^3 R_i}{4\mu_o (R_o^3 - R_i^3)} \right) / \left(\frac{R_i^4}{(2\mu_o + 3\lambda_o)(R_o^3 - R_i^3)} + \frac{R_o^3 R_i}{4\mu_o (R_o^3 - R_i^3)} - \frac{R_i}{2\mu_i + 3\lambda_i} \right) \right] P$$

In conclude, the solution of this problem is

For the inner solid elastic ball,

$$u_r^* = \frac{P^*}{2\mu + 3\lambda} r$$

For the outer layer

$$u_r = \frac{R_i^3 P^* - (2R_o^3 - R_i^3)P}{(2\mu_o + 3\lambda_o)(R_o^3 - R_i^3)} r + \frac{R_o^3 R_i^3 (P^* - P)}{4\mu_o (R_o^3 - R_i^3)} \frac{1}{r^2}$$

Here

$$P^* = \left[\left(\frac{(2R_o^3 - R_i^3)R_i}{(2\mu_o + 3\lambda_o)(R_o^3 - R_i^3)} + \frac{R_o^3 R_i}{4\mu_o (R_o^3 - R_i^3)} \right) / \left(\frac{R_i^4}{(2\mu_o + 3\lambda_o)(R_o^3 - R_i^3)} + \frac{R_o^3 R_i}{4\mu_o (R_o^3 - R_i^3)} - \frac{R_i}{2\mu_i + 3\lambda_i} \right) \right] P$$

$$\lambda = \frac{Ev}{(1+v)(1-2v)}$$

$$\mu = \frac{E}{2(1+v)}.$$

14. The stress tensor in a semi-infinite domain $x_3 > 0$ is given by $\sigma_{ij} = (\alpha x_i x_j x_3)/|x|^5$, where $|x| = (x_i x_i)^{1/2}$ and α is a constant. Determine whether there are any body forces in the domain using the equilibrium equation.

Answer:

The equilibrium equation is

$$\sigma_{ij,j} + b_i = 0$$

From the expression of stress we can get

$$\begin{aligned} \sigma_{ij,j} &= \frac{(\alpha \delta_{ij} x_j x_3)}{|x|^5} + \frac{(\alpha \delta_{jj} x_i x_3)}{|x|^5} + \frac{(\alpha x_i x_j x_3)}{|x|^7} \left(\frac{-5}{2} \right) (2x_j) \\ &= \frac{(\alpha x_i x_3)}{|x|^5} + \frac{(3\alpha x_i x_3)}{|x|^5} + \frac{(\alpha x_i x_3)}{|x|^5} - \frac{5(\alpha x_i |x|^2 x_3)}{|x|^7} \\ &= \frac{5\alpha x_i x_3}{|x|^5} - \frac{5\alpha x_i x_3}{|x|^5} \\ &= 0 \end{aligned}$$

So

$$b_i = 0 - 0 = 0$$

Body force is zero.

15. In an infinite solid with material elastic constants μ and ν , a concentrated force $P = (0, 0, 1)$ is applied at the origin. Calculate the surface force distribution on the sphere with radius 1 and the center at the origin. Integrate it over the surface for the resultant surface force.

Answer:

First we use the formula in textbook

$$u_i = \frac{P}{16\pi\mu(1-\nu)} \left[(3-4\nu) \frac{\delta_{ij}}{|x-x'|} + \frac{(x_i-x'_i)(x_j-x'_j)}{|x-x'|^3} \right] P_j$$

In this case, we get

$$u_i = \frac{P}{16\pi\mu(1-\nu)} \left(\frac{x_1 x_3}{|x|^3}, \frac{x_2 x_3}{|x|^3}, \frac{3-4\nu}{|x|} + \frac{x_3^2}{|x|^3} \right)$$

Then we can calculate strain

$$\begin{aligned} \varepsilon &= \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \\ &= \frac{P}{16\pi\mu(1-\nu)} \begin{bmatrix} \frac{\partial \left(\frac{x_1 x_3}{|x|^3} \right)}{\partial x_1} & \frac{1}{2} \left[\frac{\partial \left(\frac{x_2 x_3}{|x|^3} \right)}{\partial x_1} + \frac{\partial \left(\frac{x_1 x_3}{|x|^3} \right)}{\partial x_2} \right] & \frac{1}{2} \left[\frac{\partial \left(\frac{3-4\nu}{|x|} + \frac{x_3^2}{|x|^3} \right)}{\partial x_1} + \frac{\partial \left(\frac{x_1 x_3}{|x|^3} \right)}{\partial x_3} \right] \\ \frac{1}{2} \left[\frac{\partial \left(\frac{x_2 x_3}{|x|^3} \right)}{\partial x_1} + \frac{\partial \left(\frac{x_1 x_3}{|x|^3} \right)}{\partial x_2} \right] & \frac{\partial \left(\frac{x_2 x_3}{|x|^3} \right)}{\partial x_2} & \frac{1}{2} \left[\frac{\partial \left(\frac{3-4\nu}{|x|} + \frac{x_3^2}{|x|^3} \right)}{\partial x_2} + \frac{\partial \left(\frac{x_2 x_3}{|x|^3} \right)}{\partial x_3} \right] \\ \frac{1}{2} \left[\frac{\partial \left(\frac{3-4\nu}{|x|} + \frac{x_3^2}{|x|^3} \right)}{\partial x_1} + \frac{\partial \left(\frac{x_1 x_3}{|x|^3} \right)}{\partial x_3} \right] & \frac{1}{2} \left[\frac{\partial \left(\frac{3-4\nu}{|x|} + \frac{x_3^2}{|x|^3} \right)}{\partial x_2} + \frac{\partial \left(\frac{x_2 x_3}{|x|^3} \right)}{\partial x_3} \right] & \frac{\partial \left(\frac{3-4\nu}{|x|} + \frac{x_3^2}{|x|^3} \right)}{\partial x_3} \end{bmatrix} \\ &= \frac{P}{16\pi\mu(1-\nu)} \begin{bmatrix} \frac{x_3}{|x|^3} - \frac{3x_1^2 x_3}{|x|^5} & -\frac{3x_1^2 x_2 x_3}{|x|^5} & \frac{1}{2} \left(-\frac{3-4\nu}{|x|^3} x_1 - \frac{3x_1 x_3^2}{|x|^5} \right) \\ -\frac{3x_1^2 x_2 x_3}{|x|^5} & \frac{x_3}{|x|^3} - \frac{3x_2^2 x_3}{|x|^5} & \frac{1}{2} \left(-\frac{3-4\nu}{|x|^3} x_2 - \frac{3x_2 x_3^2}{|x|^5} \right) \\ \frac{1}{2} \left(-\frac{3-4\nu}{|x|^3} x_1 - \frac{3x_1^2 x_3^2}{|x|^5} \right) & \frac{1}{2} \left(-\frac{3-4\nu}{|x|^3} x_2 - \frac{3x_2^2 x_3^2}{|x|^5} \right) & -\frac{3-4\nu}{|x|^3} x_3 \end{bmatrix} \end{aligned}$$

According to the elasticity constitutive equation using μ, ν , the relation between stress and strain is

$$\sigma_{ij} = \frac{2\mu v}{1-2v} \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

$$\varepsilon_{kk} = \frac{P}{16\pi\mu(1-v)} \left(\frac{4x_3}{|x|^3} - \frac{3(x_1^2 + x_2^2 + x_3^2)x_3}{|x|^5} - \frac{3-4v}{|x|^3} x_3 \right)$$

$$= \frac{P}{16\pi\mu(1-v)} \left(\frac{x_3}{|x|^3} - \frac{3-4v}{|x|^3} x_3 \right)$$

$$= -\frac{P}{16\pi\mu(1-v)} \left(\frac{2-4v}{|x|^3} x_3 \right)$$

$$\sigma_{ij} = \frac{2\mu v}{1-2v} \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

$$= \frac{2\mu v}{1-2v} \frac{P}{16\pi\mu(1-v)} \begin{bmatrix} -\frac{2-4v}{|x|^3} x_3 & 0 & 0 \\ 0 & -\frac{2-4v}{|x|^3} x_3 & 0 \\ 0 & 0 & -\frac{2-4v}{|x|^3} x_3 \end{bmatrix}$$

$$+ 2\mu \frac{P}{16\pi\mu(1-v)} \begin{bmatrix} \frac{x_3}{|x|^3} - \frac{3x_1^2 x_3}{|x|^5} & -\frac{3x_1 x_2 x_3}{|x|^5} & \frac{1}{2} \left(-\frac{3-4v}{|x|^3} x_1 - \frac{3x_1^2 x_3}{|x|^5} \right) \\ -\frac{3x_1 x_2 x_3}{|x|^5} & \frac{x_3}{|x|^3} - \frac{3x_2^2 x_3}{|x|^5} & \frac{1}{2} \left(-\frac{3-4v}{|x|^3} x_2 - \frac{3x_2^2 x_3}{|x|^5} \right) \\ \frac{1}{2} \left(-\frac{3-4v}{|x|^3} x_1 - \frac{3x_1^2 x_3}{|x|^5} x_1 + \frac{x_1}{|x|^3} - \frac{3x_1 x_3^2}{|x|^5} \right) & \frac{1}{2} \left(-\frac{3-4v}{|x|^3} x_2 - \frac{3x_2^2 x_3}{|x|^5} x_2 + \frac{x_2}{|x|^3} - \frac{3x_2 x_3^2}{|x|^5} \right) & -\frac{3-4v}{|x|^3} x_3 - \frac{3x_1 x_2 x_3}{|x|^5} \end{bmatrix}$$

$$= \frac{vP}{4\pi(1-v)} \begin{bmatrix} -\frac{x_3}{|x|^3} & 0 & 0 \\ 0 & -\frac{x_3}{|x|^3} & 0 \\ 0 & 0 & -\frac{x_3}{|x|^3} \end{bmatrix}$$

$$+ \frac{P}{8\pi(1-v)} \begin{bmatrix} \frac{x_3}{|x|^3} - \frac{3x_1^2 x_3}{|x|^5} & -\frac{3x_1 x_2 x_3}{|x|^5} & \frac{1}{2} \left(-\frac{3-4v}{|x|^3} x_1 - \frac{3x_1^2 x_3}{|x|^5} \right) \\ -\frac{3x_1 x_2 x_3}{|x|^5} & \frac{x_3}{|x|^3} - \frac{3x_2^2 x_3}{|x|^5} & \frac{1}{2} \left(-\frac{3-4v}{|x|^3} x_2 - \frac{3x_2^2 x_3}{|x|^5} \right) \\ \frac{1}{2} \left(-\frac{3-4v}{|x|^3} x_1 - \frac{3x_1^2 x_3}{|x|^5} x_1 + \frac{x_1}{|x|^3} - \frac{3x_1 x_3^2}{|x|^5} \right) & \frac{1}{2} \left(-\frac{3-4v}{|x|^3} x_2 - \frac{3x_2^2 x_3}{|x|^5} x_2 + \frac{x_2}{|x|^3} - \frac{3x_2 x_3^2}{|x|^5} \right) & -\frac{3-4v}{|x|^3} x_3 - \frac{3x_1 x_2 x_3}{|x|^5} \end{bmatrix}$$

$$f_i = \sigma_{ij} n_j \quad (n_j = (x_1, x_2, x_3))$$

$$= \frac{vP}{4\pi(1-v)} \begin{bmatrix} -\frac{x_3}{|x|^3} & 0 & 0 \\ 0 & -\frac{x_3}{|x|^3} & 0 \\ 0 & 0 & -\frac{x_3}{|x|^3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{aligned}
& + \frac{P}{8\pi(1-v)} \begin{bmatrix} \frac{x_3}{|x|^3} - \frac{3x_1^2 x_3}{|x|^5} & -\frac{3x_1 x_2 x_3}{|x|^5} & \frac{1}{2} \left(-\frac{3-4v}{|x|^3} x_1 - \frac{3x_1^2 x_3}{|x|^5} \right) \\ -\frac{3x_1 x_2 x_3}{|x|^5} & \frac{x_3}{|x|^3} - \frac{3x_2^2 x_3}{|x|^5} & \frac{1}{2} \left(-\frac{3-4v}{|x|^3} x_2 - \frac{3x_2^2 x_3}{|x|^5} \right) \\ \frac{1}{2} \left(-\frac{3-4v}{|x|^3} x_1 - \frac{3x_1^2 x_3}{|x|^5} + \frac{x_1}{|x|^3} - \frac{3x_1 x_2^2}{|x|^5} \right) & \frac{1}{2} \left(-\frac{3-4v}{|x|^3} x_2 - \frac{3x_2^2 x_3}{|x|^5} + \frac{x_2}{|x|^3} - \frac{3x_2 x_3^2}{|x|^5} \right) & -\frac{3-4v}{|x|^3} x_3 - \frac{3x_1 x_2 x_3}{|x|^5} \end{bmatrix} \\
& = \frac{vP}{4\pi(1-v)} \begin{bmatrix} -\frac{x_1 x_3}{|x|^3} \\ -\frac{x_2 x_3}{|x|^3} \\ -\frac{x_3^2}{|x|^3} \end{bmatrix} + \frac{P}{8\pi(1-v)} \begin{bmatrix} \frac{x_1 x_3}{|x|^3} - \frac{3x_1^3 x_3}{|x|^5} - \frac{3x_1 x_2^2 x_3}{|x|^5} + \frac{1}{2} \left(-\frac{3-4v}{|x|^3} x_1 x_3 - \frac{3x_1^3}{|x|^5} x_1 + \frac{x_1 x_3}{|x|^3} - \frac{3x_1 x_3^4}{|x|^5} \right) \\ \frac{x_2 x_3}{|x|^3} - \frac{3x_2^3 x_3}{|x|^5} - \frac{3x_2^2 x_2 x_3}{|x|^5} + \frac{1}{2} \left(-\frac{3-4v}{|x|^3} x_2 x_3 - \frac{3x_2^3}{|x|^5} x_2 + \frac{x_2 x_3}{|x|^3} - \frac{3x_2 x_3^4}{|x|^5} \right) \\ \frac{1}{2} \left(-\frac{3-4v}{|x|^3} - \frac{3x_3^2}{|x|^3} + \frac{1}{|x|} - \frac{3x_3^2}{|x|^3} \right) - \frac{1}{2} \frac{3-4v}{|x|^3} x_3^2 + \frac{3x_3^2}{2|x|^3} \end{bmatrix} \\
& = \frac{vP}{4\pi(1-v)} \begin{bmatrix} -\frac{x_1 x_3}{|x|^3} \\ -\frac{x_2 x_3}{|x|^3} \\ -\frac{x_3^2}{|x|^3} \end{bmatrix} + \frac{P}{8\pi(1-v)} \begin{bmatrix} -\frac{(3-2v)x_1 x_3}{|x|^3} \\ -\frac{(3-2v)x_2 x_3}{|x|^3} \\ -\frac{1-2v}{|x|} - \frac{3-2v}{|x|^3} x_3^2 \end{bmatrix} \\
& = \frac{P}{8\pi(1-v)} \begin{bmatrix} -\frac{2vx_1 x_3}{|x|^3} - \frac{(3-2v)x_1 x_3}{|x|^3} \\ -\frac{2vx_2 x_3}{|x|^3} - \frac{(3-2v)x_2 x_3}{|x|^3} \\ -\frac{2vx_3^2}{|x|^3} - \frac{1-2v}{|x|} - \frac{3-2v}{|x|^3} x_3^2 \end{bmatrix} \\
& = \frac{P}{8\pi(1-v)} \begin{bmatrix} -\frac{3x_1 x_3}{|x|^3} \\ -\frac{3x_2 x_3}{|x|^3} \\ -\frac{1-2v}{|x|} - \frac{3x_3^2}{|x|^3} \end{bmatrix}
\end{aligned}$$

For the resultant surface force, they are

$$f = \int_{\partial\Omega} f_i$$

$$\begin{aligned}
& = \int_{\partial\Omega} \frac{P}{8\pi(1-v)} \begin{bmatrix} \frac{3x_1 x_3}{|x|^3} \\ \frac{3x_2 x_3}{|x|^3} \\ -\frac{1-2v}{|x|} + \frac{3x_3^2}{|x|^3} \end{bmatrix}_i \\
& = \frac{P}{8\pi(1-v)} \begin{bmatrix} 0 \\ 0 \\ - (1-2v) \int_{a=0}^{\pi} 2\pi \int_{\theta=0}^{\pi} \theta = \pi \int \sin \theta d\theta da - 3 \int_{a=0}^{\pi} 2\pi \int_{\theta=0}^{\pi} \theta = \pi \int \cos^2 \theta \sin \theta d\theta da \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \frac{P}{8\pi(1-\nu)} \begin{bmatrix} 0 \\ 0 \\ -(1-2\nu) 2\pi [-\cos \theta]_0^\pi - 3 \cdot 2\pi \left[\frac{-\cos^3 \theta}{3} \right]_0^\pi \end{bmatrix} \\
&= \frac{P}{8\pi(1-\nu)} \begin{bmatrix} 0 \\ 0 \\ -(1-2\nu) 4\pi - 4\pi \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ -P \end{bmatrix}
\end{aligned}$$

The first two items are zero because symmetry.

The result is correct in which the force around the sphere balance the concentrated force.

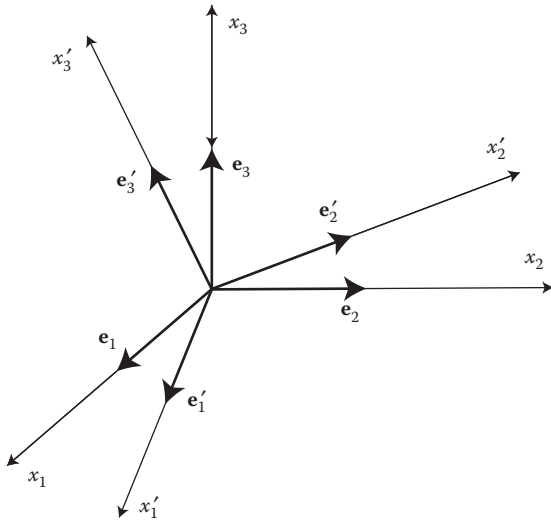


Figure 2.1 Three-dimensional Cartesian coordinate and coordinate rotation.

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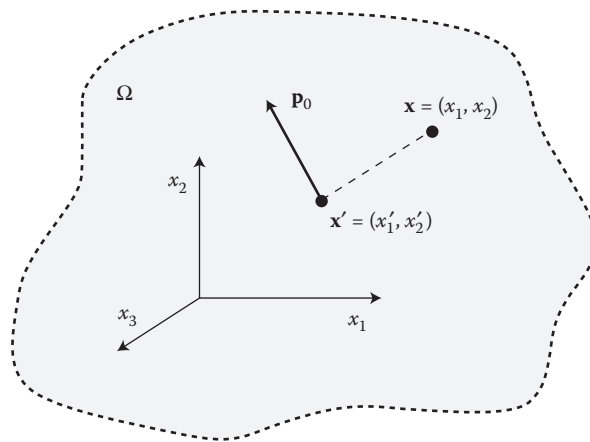


Figure 2.2 Green's function.

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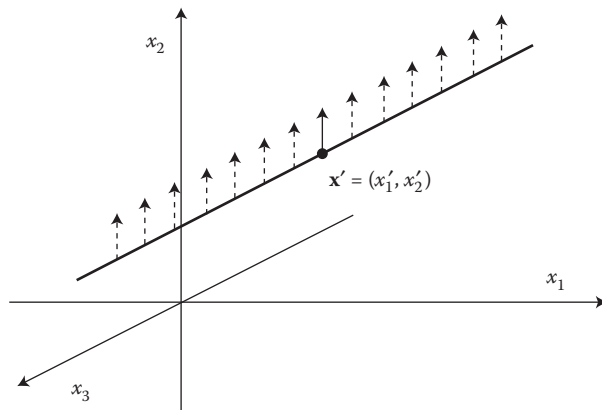


Figure 2.3 Dimension reduction of Green's function.

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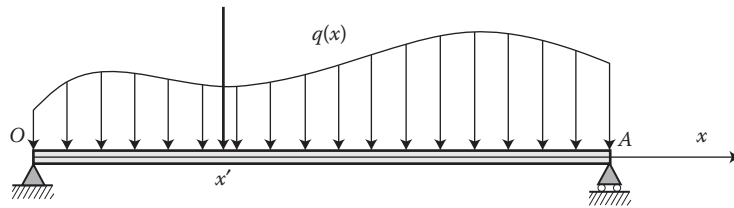


Figure 2.4 Straight beam under distribution loading.

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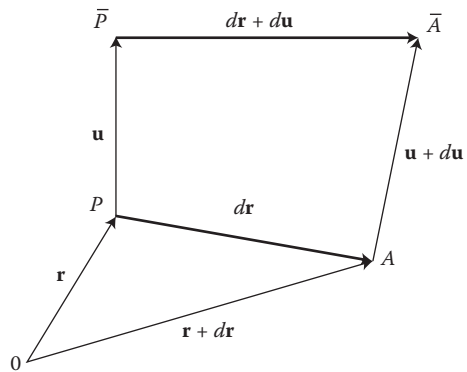


Figure 2.5 Deformation of an infinitesimal line element.

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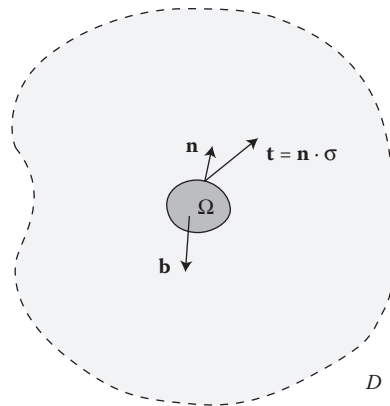


Figure 2.6 The equilibrium of an arbitrary domain.

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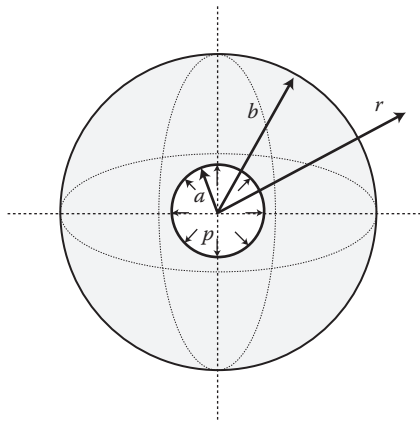


Figure 2.7 A ball subjected to a uniform inner pressure p .

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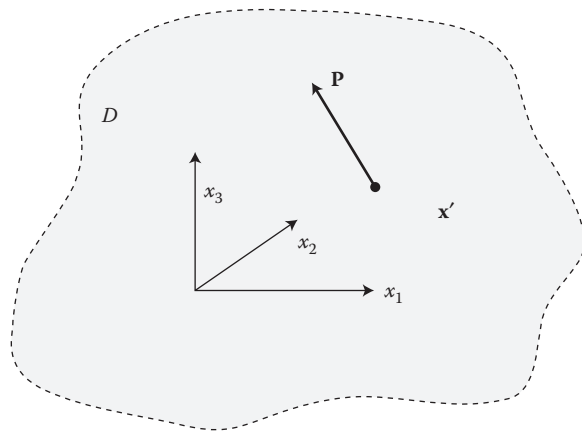


Figure 2.8 Concentrated force acting on an infinite domain.

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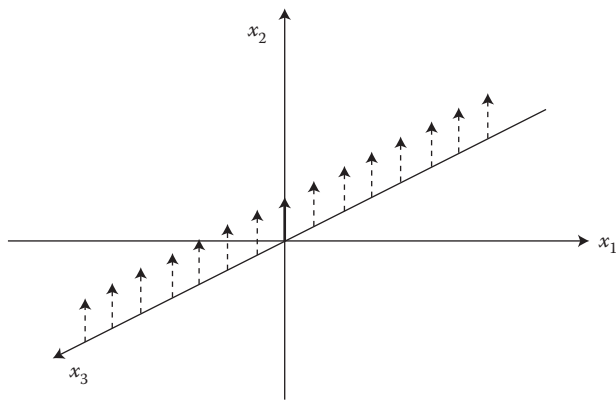


Figure 2.9 Dimension reduction of Kelvin's solution.

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