

***Stochastic Processes From Applications to Theory 1st
edition Moral Solutions Manual***

SKU: 9781498701839-SOLUTIONS

Chapter 2

Solution to exercise 11:

- For the Bernoulli distribution $p(x) = p^x(1-p)^{1-x}$, with $p \in [0, 1]$ and $x \in \{0, 1\}$, we have

$$\varphi(s) = \mathbb{E}(s^X) = p s + (1-p) s^0 = ps + (1-p).$$

- For the Binomial distribution $p(x) = \binom{n}{x} p^x(1-p)^{n-x}$, with $p \in [0, 1]$ and $x \in \{0, \dots, n\}$ for some $n \in \mathbb{N}$, we have

$$\begin{aligned} \varphi(s) &= \sum_{0 \leq x \leq n} \binom{n}{x} s^x p^x (1-p)^{n-x} \\ &= \sum_{0 \leq x \leq n} \binom{n}{x} (sp)^x (1-p)^{n-x} = ((1-p) + ps)^n. \end{aligned}$$

- For the Poisson distribution $p(x) = e^{-\lambda} \lambda^x / x!$, with $\lambda > 0$ and $x \in \mathbb{N}$, we have

$$\begin{aligned} \varphi(s) &= e^{-\lambda} \sum_{x \geq 0} s^x \lambda^x / x! \\ &= e^{-\lambda} \sum_{x \geq 0} (s\lambda)^x / x! = e^{-\lambda + s\lambda} = e^{-\lambda(1-s)}. \end{aligned}$$

- For the Geometric distribution $p(x) = (1-p)^{x-1}p$, with $\lambda > 0$ and $x \in \mathbb{N} - \{0\}$, we have

$$\begin{aligned} \varphi(s) &= p \sum_{x \geq 1} s^x (1-p)^{x-1} \\ &= ps \sum_{x \geq 1} s^{x-1} (1-p)^{x-1} = ps \sum_{x \geq 0} (s(1-p))^x = ps / (1 - s(1-p)). \end{aligned}$$

This ends the proof of the exercise. ■

Solution to exercise 12:

By construction, we have

$$\begin{aligned} \mathbb{E}(N_{n+1}) &= \mathbb{E} \left(\mathbb{E} \left(\sum_{1 \leq i \leq N_n} X_n^i \mid N_n \right) \right) \\ &= \mathbb{E} \left(\sum_{1 \leq i \leq N_n} \mathbb{E}(X_n^i \mid N_n) \right) = \mathbb{E}(N_n) m. \end{aligned}$$

$$\begin{aligned}
\text{Var}(N_{n+1}) &= \mathbb{E}(N_{n+1}^2) - (\mathbb{E}(N_{n+1}))^2 \\
&= \mathbb{E} \left(\mathbb{E} \left(\left(\sum_{1 \leq i \leq N_n} X_n^i \right)^2 \mid N_n \right) \right) - (\mathbb{E}(N_{n+1}))^2.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\mathbb{E} \left(\left(\sum_{1 \leq i \leq N_n} X_n^i \right)^2 \mid N_n \right) &= N_n \mathbb{E}(X^2) + N_n(N_n - 1) (\mathbb{E}(X))^2 \\
&= N_n \text{Var}(X) + N_n^2 m^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
\text{Var}(N_{n+1}) &= \mathbb{E}(N_n) \text{Var}(X) + \mathbb{E}(N_n^2) m^2 - (\mathbb{E}(N_{n+1}))^2 \\
&= \mathbb{E}(N_n) \text{Var}(X) + \text{Var}(N_n) m^2 + [\mathbb{E}(N_n) m]^2 - (\mathbb{E}(N_{n+1}))^2 \\
&= m^2 \text{Var}(N_n) + \mathbb{E}(N_n) \text{Var}(X).
\end{aligned}$$

We conclude that

$$\begin{aligned}
\text{Var}(N_{n+1}) &= m^2 [m^2 \text{Var}(N_{n-1}) + \mathbb{E}(N_{n-1}) \text{Var}(X)] + \mathbb{E}(N_n) \text{Var}(X) \\
&= m^4 \text{Var}(N_{n-1}) + [m^2 \mathbb{E}(N_{n-1}) + \mathbb{E}(N_n)] \text{Var}(X) \\
&= m^6 \text{Var}(N_{n-2}) + [m^4 \mathbb{E}(N_{n-2}) + m^2 \mathbb{E}(N_{n-1}) + \mathbb{E}(N_n)] \text{Var}(X) \\
&= \dots \\
&= m^{2(n+1)} \text{Var}(N_0) + \text{Var}(X) \sum_{0 \leq k \leq n} m^{2k} \mathbb{E}(N_{n-k}),
\end{aligned}$$

so that

$$\text{Var}(N_{n+1}) = m^{2(n+1)} \text{Var}(N_0) + \text{Var}(X) m^n (\mathbb{E}(N_0))^n \sum_{0 \leq k \leq n} (m/\mathbb{E}(N_0))^k.$$

When $N_0 = 1$ we have $\text{Var}(N_0) = 0$ and $\mathbb{E}(N_0) = 1$. In this case, we have

$$\text{Var}(N_n) = \text{Var}(X) m^{n-1} \sum_{0 \leq k < n} m^k = \begin{cases} n \text{Var}(X) & \text{when } m = 1 \\ \text{Var}(X) m^{n-1} \frac{m^n - 1}{m - 1} & \text{when } m \neq 1. \end{cases}$$

This ends the proof of the exercise. ■

Solution to exercise 13:

We have

$$\begin{aligned}
\varphi_n(s) &:= \mathbb{E}(\mathbb{E}(s^{N_n} \mid N_{n-1})) = \mathbb{E} \left(\prod_{1 \leq i \leq N_{n-1}} \mathbb{E}(s^{X_n^i} \mid N_{n-1}) \right) \\
&= \mathbb{E}(\mathbb{E}(s^X)^{N_{n-1}}) = \mathbb{E}(\varphi_1(s)^{N_{n-1}}) = \varphi_{n-1}(\varphi_1(s)).
\end{aligned}$$

Recalling that $0^0 = 1$, this implies that

$$\varphi_n(0) = \mathbb{E}(0^{N_n}) = 1 \times \mathbb{P}(N_n = 0) = \mathbb{P}(N_n = 0).$$

For the Bernoulli offspring distribution $p(x) = p^x q^{1-x}$, with $q := (1 - p)$, $p \in [0, 1]$ and $x \in \{0, 1\}$, we have

$$\begin{aligned} \varphi_1(s) &= q + ps \\ \varphi_2(s) &= \varphi_1(q + ps) = q + p(q + ps) = q(1 + p) + p^2 s \\ \varphi_3(s) &= \varphi_2(q + ps) = q(1 + p) + p^2(q + ps) = q(1 + p + p^2) + p^3 s \\ &\dots = \dots \\ \varphi_n(s) &= q(1 + p + p^2 + \dots + p^{n-1}) + p^n s = \frac{q}{1 - p} (1 - p^n) + p^n s = (1 - p^n) + p^n s. \end{aligned}$$

The last assertion follows from the fact that $\varphi_n(s) = (1 - p^n) + p^n s$ is the moment generating function of a Bernoulli random variable N_n with parameter p^n ; that is, we have that

$$\mathbb{P}(N_n = 1) = p^n \quad \text{and} \quad \mathbb{P}(N_n = 0) = 1 - p^n.$$

This ends the proof of the exercise. ■

Solution to exercise 14:

We set $g_0^i = (g_0^i(j))_{j \in S}$. In this notation, we have

$$\mathbb{E} \left(\sum_{1 \leq i \leq N_1} f(\widehat{\xi}_0^i) \mid g_0^i, \xi_0 \right) = \sum_{1 \leq i \leq N_0} g_0^i(\xi_0^i) f(\xi_0^i).$$

This implies that

$$\mathbb{E} \left(\sum_{1 \leq i \leq N_1} f(\widehat{\xi}_0^i) \mid \xi_0 \right) = \sum_{1 \leq i \leq N_0} G(\xi_0^i) f(\xi_0^i)$$

and therefore

$$\mathbb{E} \left(\sum_{1 \leq i \leq N_1} f(\widehat{\xi}_0^i) \right) = \sum_{1 \leq i \leq N_0} \eta_0(Gf) = N_0 \eta_0(Gf). \quad (30.18)$$

In the same vein, we have

$$\mathbb{E} \left(\sum_{1 \leq i \leq N_1} f(\xi_1^i) \mid N_1, \widehat{\xi}_0 \right) = \sum_{1 \leq i \leq N_1} \mathbb{E} \left(f(\xi_1^i) \mid \widehat{\xi}_0^i \right) = \sum_{1 \leq i \leq N_1} M(f)(\widehat{\xi}_0^i).$$

Using (30.18), we readily deduce that

$$\mathbb{E} \left(\sum_{1 \leq i \leq N_1} f(\xi_1^i) \right) = \mathbb{E} \left(\sum_{1 \leq i \leq N_1} M(f)(\widehat{\xi}_0^i) \right) = N_0 \eta_0(GM(f)). \quad (30.19)$$

In much the same way, if we set $g_1^i = (g_1^i(j))_{j \in S}$ then we have

$$\mathbb{E} \left(\sum_{1 \leq i \leq N_2} f(\widehat{\xi}_1^i) \mid g_1^i, \xi_1 \right) = \sum_{1 \leq i \leq N_1} g_1^i(\xi_1^i) f(\xi_1^i).$$

This implies that

$$\mathbb{E} \left(\sum_{1 \leq i \leq N_2} f(\hat{\xi}_1^i) \mid \xi_1 \right) = \sum_{1 \leq i \leq N_1} G(\xi_1^i) f(\xi_1^i).$$

Using (30.18), we readily deduce that

$$\mathbb{E} \left(\sum_{1 \leq i \leq N_2} f(\hat{\xi}_1^i) \right) = \mathbb{E} \left(\sum_{1 \leq i \leq N_1} G(\xi_1^i) f(\xi_1^i) \right) = N_0 \eta_0(Q(Gf)). \quad (30.20)$$

Arguing as above we have

$$\mathbb{E} \left(\sum_{1 \leq i \leq N_2} f(\xi_2^i) \mid N_2, \hat{\xi}_1 \right) = \sum_{1 \leq i \leq N_2} \mathbb{E} \left(f(\xi_2^i) \mid \hat{\xi}_1^i \right) = \sum_{1 \leq i \leq N_2} M(f)(\hat{\xi}_1^i).$$

Using (30.20) we deduce that

$$\mathbb{E} \left(\sum_{1 \leq i \leq N_2} f(\xi_2^i) \right) = \mathbb{E} \left(\sum_{1 \leq i \leq N_2} M(f)(\hat{\xi}_1^i) \right) = N_0 \eta_0(Q(GM(f))) = N_0 \eta_0(Q^2(f)). \quad (30.21)$$

The last assertion is proved using induction. This ends the proof of the exercise. ■

Solution to exercise 15:

By construction, we have

$$\mathbb{P}(X_{n+1} = i \mid X_1, \dots, X_n) = \frac{n}{n+\alpha} \frac{1}{n} \sum_{1 \leq p \leq n} 1_{X_p}(i) + \frac{\alpha}{n+\alpha} \mu(i). \quad (30.22)$$

The number of different tables occupied by the first n customers is defined by

$$T_n := \sum_{1 \leq p \leq n} \epsilon_p$$

where ϵ_n stands for a sequence of independent Bernoulli random variables with distribution

$$\mathbb{P}(\epsilon_n = 1) = 1 - \mathbb{P}(\epsilon_n = 0) = \frac{\alpha}{\alpha + (n-1)}.$$

This implies that

$$\sum_{1 \leq p < n} \int_p^{p+1} \frac{dt}{1 + (t/\alpha)} \leq \mathbb{E}(T_n) = \sum_{0 \leq p < n} \frac{\alpha}{\alpha + p} \leq \sum_{1 \leq p < n} \int_{p-1}^p \frac{dy}{1 + (t/\alpha)}.$$

We conclude that

$$\int_1^n \frac{dt}{1 + (t/\alpha)} = \alpha \log \left(\frac{\alpha + n}{\alpha + 1} \right) \leq \mathbb{E}(T_n) \leq \int_0^{n-1} \frac{dt}{1 + (t/\alpha)} = \alpha \log (1 + (n-1)/\alpha).$$

The formula

$$\mathbb{P}(X_{n+1} = i \mid X_1, \dots, X_n) = \frac{\alpha \mu(i) + V_n(i)}{\alpha + n} \quad \text{with} \quad V_n(i) = \sum_{1 \leq p \leq n} 1_{X_p}(i)$$

is a direct consequence of (30.22).

This ends the proof of the exercise. $\blacksquare$

Solution to exercise 16:

For each $s \in S$ and $x = (x_1, \dots, x_{n+1})$ we let $t_k(s, x) \in \{1, \dots, n+1\}$, with $k = 1, \dots, v_{n+1}(s)$ be the times at which $x_{t_k(s, x)} = s$. In this notation, we have

$$\begin{aligned} \mathbb{P}(X_1 = x_1, \dots, X_{n+1} = x_{n+1}) &= \prod_{s \in S} \prod_{0 \leq k < v_{n+1}(s)} \frac{\alpha \mu(s) + k}{\alpha + (t_k(s, x) - 1)} \\ &= \left[\prod_{0 \leq t \leq n} \frac{1}{\alpha + t} \right] \prod_{s \in S} \prod_{0 \leq k < v_{n+1}(s)} (\alpha \mu(s) + k). \end{aligned}$$

In the last assertion we have used the fact that $\mathcal{T}(s, x) := \{t_k(s, x), k = 1, \dots, v_{n+1}(s)\}$, with $s \in S$ is a partition of the set $\{1, \dots, n+1\}$

$$\cup_{s \in S} \mathcal{T}(s, x) = \{1, \dots, n+1\}.$$

The formula (2.6) coincides with (4.9) when $a_s = \alpha \mu(s)$. Following the arguments described on page 79, we conclude that $(X_i)_{i \geq 1}$ can be interpreted as a sequence of independent random variables on the set $S := \{1, \dots, d\}$ with probability distribution given by (2.7). By the law of large numbers, given U check that $\frac{1}{n} \sum_{1 \leq p \leq n} 1_{X_p}(i)$ converges almost surely to U_i , as $n \uparrow \infty$. In addition, we have

$$\mathbb{E} \left(\frac{1}{n} \sum_{1 \leq p \leq n} 1_{X_p}(i) \mid U \right) = U_i \quad \text{and} \quad \text{Var} \left(\frac{1}{n} \sum_{1 \leq p \leq n} 1_{X_p}(i) \mid U \right) = \frac{1}{n} (U_i(1 - U_i)).$$

This ends the proof of the exercise. $\blacksquare$

Solution to exercise 17:

The first assertion is immediate. In addition, we have that

$$\begin{aligned} S_n(f) &= \frac{n}{n+1} \left(\frac{1}{n} \sum_{k=0}^{n-1} f(X_k) + \frac{1}{n} f(X_n) \right) \\ &= \frac{n}{n+1} S_{n-1}(f) + \frac{1}{n+1} f(X_n). \end{aligned}$$

By construction, we have

$$\mathbb{E}(f(X_{n+1}) \mid X_0, \dots, X_n) = \epsilon S_n(f) + (1 - \epsilon) \mu(f).$$

In other words, this yields

$$\mathbb{E}([f(X_{n+1}) - \mu(f)] \mid X_0, \dots, X_n) = \epsilon S_n([f - \mu(f)]) .$$

Thus, for any function f such that $\mu(f) = 0$, we have

$$\mathbb{E}(f(X_{n+1}) \mid X_0, \dots, X_n) = \epsilon S_n(f) \quad \Rightarrow \quad \eta_{n+1}(f) = \epsilon \bar{S}_n(f) .$$

Recalling that

$$\mathbb{E}(f(X_{n+1})) = \mathbb{E}(\mathbb{E}(f(X_{n+1}) \mid X_0, \dots, X_n))$$

we prove that

$$\begin{aligned}
 \bar{S}_n(f) &= \frac{n}{n+1} \bar{S}_{n-1}(f) + \frac{1}{n+1} \mathbb{E}(f(X_n)) \\
 &= \frac{n+\epsilon}{n+1} \times \bar{S}_{n-1}(f) \\
 &= \frac{n+\epsilon}{n+1} \times \frac{(n-1)+\epsilon}{(n-1)+1} \times \bar{S}_{n-2}(f) = \dots = \left[\prod_{k=1}^n \frac{k+\epsilon}{k+1} \right] \times \mathbb{E}(f(X_0)).
 \end{aligned}$$

We observe that

$$k \leq t \leq k+1 \Rightarrow \log \left(1 - \frac{(1-\epsilon)}{k} \right) \leq \log \left(1 - \frac{(1-\epsilon)}{t} \right) \leq \log \left(1 - \frac{(1-\epsilon)}{k+1} \right).$$

This implies that

$$\sum_{1 \leq k \leq n} \int_k^{k+1} \log \left(1 - \frac{(1-\epsilon)}{t} \right) dt \leq \log \alpha_\epsilon(n)$$

and

$$\log \alpha_\epsilon(n) \leq \sum_{1 \leq k \leq n} \int_{k+1}^{k+2} \log \left(1 - \frac{(1-\epsilon)}{t} \right) dt.$$

This ends the proof of (2.8). Using the estimates

$$\forall x \in [0, 1[\quad -\frac{x}{1-x} \leq \log(1-x) \leq -x$$

we check that

$$\int_2^{n+2} \log \left(1 - \frac{(1-\epsilon)}{t} \right) dt \leq -(1-\epsilon) \log(1+n/2)$$

and

$$\int_1^{n+1} \log \left(1 - \frac{(1-\epsilon)}{t} \right) dt \geq -(1-\epsilon) \log(1+n/\epsilon).$$

The end of the proof of the exercise is immediate. ■

Solution to exercise 18:

By construction, we have

$$\begin{aligned}
 M(f)(i) = \epsilon K(f)(i) + (1-\epsilon) \nu(f) &\Rightarrow [M(f)(i) - M(f)(j)] = \epsilon [K(f)(i) - K(f)(j)] \\
 &\Rightarrow \text{osc}(M(f)) \leq \epsilon \text{osc}(f).
 \end{aligned}$$

Assuming that $\text{osc}(M^n(f)) \leq \epsilon^n \text{osc}(f)$ is true at rank n , we have

$$\text{osc}(M^{n+1}(f)) = \text{osc}(M^n(M(f))) \leq \epsilon^n \text{osc}(M(f)) \leq \epsilon^{n+1} \text{osc}(f).$$

Recall that

$$f = 1_k \Rightarrow M^n(f)(i) = M^n(i, k) = \mathbb{P}(X_n = k | X_0 = i).$$

The end of the proof of the exercise is now clear. ■

Solution to exercise 19: The first assertion is immediate since $\frac{d\bar{W}_t}{dt} = W_t \frac{dW_t}{dt}$. To check the second one, we observe that

$$X_n = a_n X_{n-1} + b_n = \left[\prod_{p=1}^n a_p \right] X_0 + \sum_{1 \leq p \leq n} \left[\prod_{n \geq q > p} a_q \right] b_p$$

with the sequence of random variables

$$a_n = (1 - \epsilon_n) + \epsilon_n 4^{-1} = 4^{-\epsilon_n} \quad \text{and} \quad b_n = (1 - \epsilon_n) h$$

Using the fact that

$$\begin{aligned} & \text{Law}((a_1, \dots, a_{p+1}, \dots, a_n), (b_1, \dots, b_p, \dots, b_n)) \\ &= \\ & \text{Law}((a_n, \dots, a_{n-p}, \dots, a_1), (b_n, \dots, b_{n-p+1}, \dots, b_1)). \end{aligned}$$

we check that

$$\sum_{1 \leq p \leq n} \left[\prod_{n \geq q > p} a_q \right] b_p \stackrel{\text{law}}{=} \sum_{1 \leq p \leq n} [a_1 \dots a_{n-p}] b_{(n-p)+1} = \sum_{0 \leq p < n} [a_1 \dots a_p] b_{p+1}$$

The end of the proof of the exercise is now clear. ■

Solution to exercise 20:

We have

$$\begin{aligned} & \mathbb{P}(X_T = x_{\max} | X_0 = x) \\ &= \mathbb{E}(\mathbb{P}(X_T = x_{\max} | X_1) | X_0 = x) \\ &= p \underbrace{\mathbb{P}(X_T = x_{\max} | X_1 = x+1)}_{:=P(x+1)} + (1-p) \underbrace{\mathbb{P}(X_T = x_{\max} | X_1 = x-1)}_{:=P(x-1)}. \end{aligned}$$

On the other hand

$$\begin{aligned} P(x) &= pP(x) + qP(x) = pP(x+1) + qP(x-1) \\ \Rightarrow p[P(x+1) - P(x)] &= q[P(x) - P(x-1)] \\ \Rightarrow [P(x+1) - P(x)] &= \frac{q}{p}[P(x) - P(x-1)]. \end{aligned}$$

Recalling that $P(0) = 0$, this yields

$$[P(2) - P(1)] = \frac{p}{q} P(1) \Rightarrow [P(3) - P(2)] = \frac{p}{q} [P(2) - P(1)] = \left(\frac{p}{q}\right)^2 P(1).$$

By a simple induction w.r.t. x we find that

$$\begin{aligned} [P(x) - P(x-1)] &= \left(\frac{p}{q}\right)^{x-1} P(1) \\ \Rightarrow [P(x+1) - P(x)] &= \frac{p}{q} [P(x) - P(x-1)] = \left(\frac{p}{q}\right)^x P(1). \end{aligned}$$

On the other hand, we have

$$P(x+1) = [P(x+1) - P(0)] = \sum_{0 \leq y \leq x} [P(y+1) - P(y)] = P(1) \sum_{0 \leq y \leq x} \left(\frac{p}{q}\right)^y.$$

We end the proof using the fact that

$$x = x_{\max} - 1 \Rightarrow P(x+1) = P(x_{\max}) = 1 = P(1) \sum_{0 \leq y < x_{\max}} \left(\frac{p}{q}\right)^y$$

so that

$$P(1) = 1 / \sum_{0 \leq y < x_{\max}} \left(\frac{p}{q}\right)^y.$$

This implies that

$$P(x+1) = \frac{\sum_{0 \leq y \leq x} \left(\frac{p}{q}\right)^y}{\sum_{0 \leq y < x_{\max}} \left(\frac{p}{q}\right)^y} = \begin{cases} \frac{1 - \left(\frac{q}{p}\right)^{x+1}}{1 - \left(\frac{q}{p}\right)^{x_{\max}}} & \text{if } p \neq q \\ \frac{(x+1)}{x_{\max}} & \text{if } p = q. \end{cases}$$

This ends the proof of the exercise. ■