

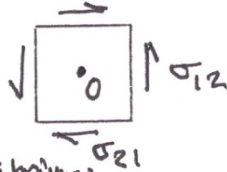
***Principles of Composite Material Mechanics 4th
edition Gibson Solutions Manual***

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Chapter 2

2.1

For two dimensional plane stress in the 12 plane, the shear stresses act as shown on the element below

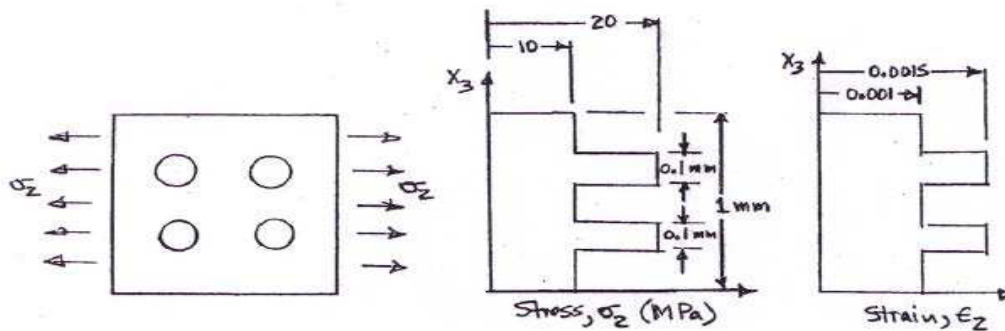


For static equilibrium:

$\sum M_o = 0$ at the center of the element implies that

$\sigma_{12} = \sigma_{21}$. Similarly, in general $\sigma_{ij} = \sigma_{ji}$ when $i \neq j$

2.2



The average transverse stress is

$$\bar{\sigma}_2 = \frac{\int_A \sigma_2 dA}{\int_A dA} = \frac{20 \left[4 \left(\frac{\pi}{4} \right) (0.1)^2 + 10 \left[(1)^2 - 4 \left(\frac{\pi}{4} \right) (0.1)^2 \right] \right]}{(1)^2}$$

$$= 10.314 \text{ MPa}$$

The average transverse strain is

$$\bar{\epsilon}_2 = \frac{\int_A \epsilon_2 dA}{\int_A dA} = \frac{(0.0015) 4 \left(\frac{\pi}{4} \right) (0.1)^2 + 0.001 - 4 \left(\frac{\pi}{4} \right) (0.1)^2}{(1)^2}$$

$$= 0.001$$

the effective transverse modulus is

$$E_2 = \frac{\bar{\sigma}_2}{\bar{\epsilon}_2} = \frac{10.314}{0.001} = 10314 \text{ MPa} = 10.314 \text{ GPa}$$

2.3

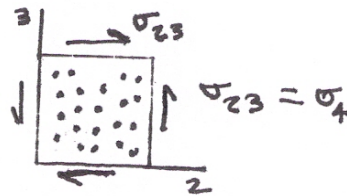
Due to the interchangeability of subscripts 2 and 3 for isotropy in the 2-3 plane,

$$C_{22} = C_{33} \text{ and } C_{12} = C_{13}$$

$$\sigma_{13} = \sigma_{31} = \sigma_5 = \sigma_{12} = \sigma_{21} = \sigma_6$$

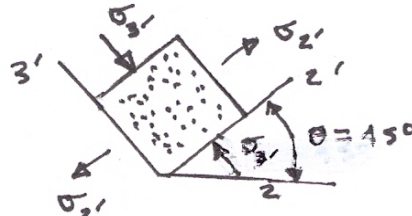
$$\gamma_{13} = \gamma_{31} = \epsilon_5 = \gamma_{12} = \gamma_{21} = \epsilon_6$$

It can be shown that C_{44} is not independent of C_{22} and C_{23} by considering an element subjected to a pure shear stress $\sigma_{23} = \sigma_4$ as shown below:



From Eqs 2.4 and 2.16, $\sigma_4 = C_{44} \epsilon_4$

From mechanics of materials, the corresponding stresses on an element oriented at $\theta = 45^\circ$ to the 2-3 axes are obtained from stress transformations as



$$\sigma_{2'} = 2 \sigma_{23} \cos 45^\circ \sin 45^\circ = \sigma_{23} = \sigma_4$$

$$\sigma_{3'} = -2 \sigma_{23} \cos 45^\circ \sin 45^\circ = -\sigma_{23} = -\sigma_4$$

Similarly, the strain transformations are

$$\epsilon_{2'} = 2 \frac{\gamma_{23}}{2} \cos 45^\circ \sin 45^\circ = \frac{\gamma_{23}}{2} = \epsilon_{23}$$

$$\epsilon_{3'} = -2 \frac{\gamma_{23}}{2} \cos 45^\circ \sin 45^\circ = -\frac{\gamma_{23}}{2} = -\epsilon_{23}$$

Using Eqs 2.4 and 2.17 for the 2'-3' axes and noting that

$$C_{22}' = C_{22} \text{ and } C_{23}' = C_{23} \text{ due to transverse isotropy,}$$

$$\sigma_{2'} = C_{22} \epsilon_{2'} + C_{23} \epsilon_{3'} = \frac{\gamma_{23}}{2} (C_{22} - C_{23})$$

But recall that $\sigma_{2'} = \sigma_{23} = \frac{\gamma_{23}}{2} (C_{22} - C_{23}) = C_{44} \epsilon_4 = C_{44} \frac{\gamma_{23}}{2}$

Therefore,

$$C_{44} = \frac{1}{2} (C_{22} - C_{23})$$

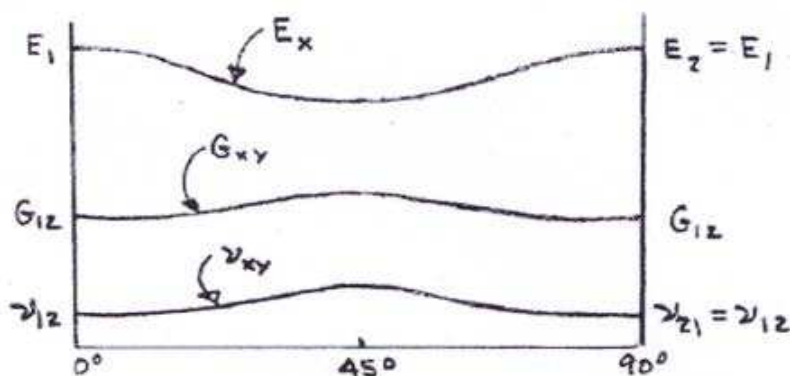
2.4

(a) For the balanced orthotropic, or square symmetric lamina shown in Fig. 2.9 we must have $E_1 = E_2$ and $\nu_{12} = \nu_{21}$, so that the lamina stress-strain relationships will be of the form

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_1} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

therefore, this material only has three independent elastic constants.

(b)



2.5

From Eqs. (2.24), we have

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

Inverting the compliance matrix, we have

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

where the stiffnesses Q_{ij} are given by

$$[Q] = [S]^{-1} = \frac{[a]^T}{|S|}$$

the cofactor matrix $[a]$ consists of elements

$$a_{ij} = (-1)^{i+j} M_{ji}$$

continued

2.5 continued

where the minor element M_{ij} is found by deleting row i and column j from $[S]$ and taking the determinant of the remaining terms, the determinant $|S|$ is given by

$$|S| = \sum_{i=1}^n S_{ij} a_{ij} = (S_{11} S_{22} - S_{12}^2) S_{66}$$

the cofactor matrix is

$$[a] = \begin{bmatrix} S_{22} S_{66} & -S_{12} S_{66} & 0 \\ -S_{12} S_{66} & S_{11} S_{66} & 0 \\ 0 & 0 & S_{11} S_{22} - S_{12}^2 \end{bmatrix} = [a]^T$$

Finally, the stiffness matrix is

$$[Q] = [S]^{-1} = \frac{[a]^T}{|S|} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}$$

where $Q_{11} = \frac{S_{22}}{S_{11} S_{22} - S_{12}^2}$; $Q_{12} = -\frac{S_{12}}{S_{11} S_{22} - S_{12}^2}$

$Q_{22} = \frac{S_{11}}{S_{11} S_{22} - S_{12}^2}$; $Q_{66} = \frac{1}{S_{66}}$

By substituting Eqs. (2.25) in the above expressions, we can show that

$$Q_{11} = \frac{E_1}{1 - \nu_{12} \nu_{21}} ; \quad Q_{12} = \frac{\nu_{12} E_2}{1 - \nu_{12} \nu_{21}}$$

$$Q_{22} = \frac{E_2}{1 - \nu_{12} \nu_{21}} ; \quad Q_{66} = G_{12}$$

2.6

From Table 2.2, for AS/3501 carbon/epoxy,

$$E_1 = 138 \text{ GPa}, E_2 = 9 \text{ GPa}, G_{12} = 6.9 \text{ GPa}, \nu_{12} = 0.3$$

$$\nu_{21} = \nu_{12} \frac{E_2}{E_1} = (0.3) \frac{9}{138} = 0.0196$$

From Eqs. (2.27):

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}} = \frac{138}{1 - 0.3(0.0196)} = 138.86 \text{ GPa}$$

$$Q_{12} = \frac{\nu_{12} E_2}{1 - \nu_{12}\nu_{21}} = \frac{(0.3)(9)}{1 - 0.3(0.0196)} = 2.716 \text{ GPa} = Q_{21}$$

$$Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}} = \frac{9}{1 - 0.3(0.0196)} = 9.05 \text{ GPa}$$

$$Q_{66} = G_{12} = 6.9 \text{ GPa}$$

so that

$$[Q] = \begin{bmatrix} 138.8 & 2.716 & 0 \\ 2.716 & 9.05 & 0 \\ 0 & 0 & 6.9 \end{bmatrix} \text{ GPa}$$

From Eqs. (2.25):

$$S_{11} = \frac{1}{E_1} = \frac{1}{138} = 0.00725 \text{ (GPa)}^{-1}$$

$$S_{12} = -\frac{\nu_{12}}{E_1} = -\frac{0.3}{138} = -0.00217 \text{ (GPa)}^{-1} = S_{21}$$

$$S_{22} = \frac{1}{E_2} = \frac{1}{9} = 0.111 \text{ (GPa)}^{-1}$$

$$S_{66} = \frac{1}{G_{12}} = \frac{1}{6.9} = 0.145 \text{ (GPa)}^{-1}$$

so that

$$[S] = \begin{bmatrix} 0.00725 & -0.00217 & 0 \\ -0.00217 & 0.111 & 0 \\ 0 & 0 & 0.145 \end{bmatrix} \text{ (GPa)}^{-1}$$

$$\text{and } [S] = [Q]^{-1}$$

2.7

Substituting the given conditions $\sigma_2 = \sigma_3 = \sigma$, $\epsilon_1 = 0$, $\epsilon_2 = \epsilon_3 = \epsilon$ into the 3-D stress-strain relationships for a specially orthotropic, transversely isotropic material (i.e., Egs 2.22 with $E_2 = E_3$, $\nu_{21} = \nu_{31}$, etc.), we find that

$$\epsilon_1 = 0 = \frac{\sigma_1}{E_1} - \frac{\nu_{12}\sigma_2}{E_1} - \frac{\nu_{12}\sigma_3}{E_1} = \frac{\sigma_1}{E_1} - \frac{2\nu_{12}\sigma}{E_1}$$

$$\text{or } \sigma_1 = 2\nu_{12}\sigma$$

Furthermore,

$$\epsilon_2 = \epsilon = -\frac{\nu_{12}}{E_1}\sigma_1 + \frac{\sigma_2}{E_2} - \frac{\nu_{23}\sigma_3}{E_2}$$

so that

$$\epsilon_2 = \epsilon = -\frac{\nu_{12}}{E_1}(2\nu_{12}\sigma) + \frac{\sigma}{E_2} - \frac{\nu_{23}\sigma}{E_2}$$

$$\text{or } \sigma = 2K_{23}\epsilon$$

$$\text{where } 2K_{23} = \frac{1}{-\frac{2\nu_{12}^2}{E_1} + \frac{1}{E_2} - \frac{\nu_{23}}{E_2}}$$

Rearranging, we get

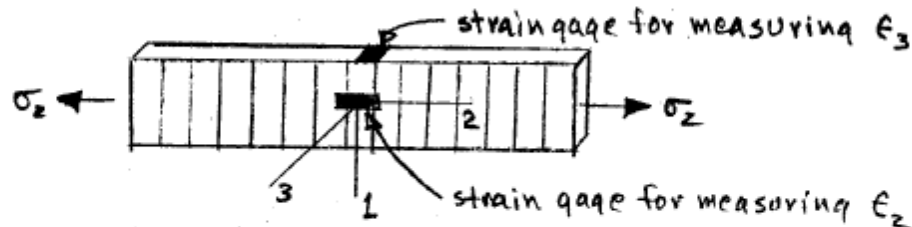
$$\frac{1}{K_{23}} = -\frac{4\nu_{12}^2}{E_1} + \frac{2}{E_2} - \frac{2\nu_{23}}{E_2}$$

$$\frac{1}{K_{23}} = -\frac{4\nu_{12}^2}{E_1} + \frac{4}{E_2} - \frac{2}{E_2}(1+\nu_{23})$$

$$\frac{1}{K_{23}} = -\frac{4\nu_{12}^2}{E_1} + \frac{4}{E_2} + \frac{1}{G_{23}} ; \text{ since } G_{23} = \frac{E_2}{2(1+\nu_{23})}$$

2.8

Conduct a uniaxial tensile test along the transverse direction by applying a stress condition $\sigma_2 \neq 0$, $\sigma_1 = \sigma_3 = \tau_{12} = \tau_{13} = \tau_{23} = 0$.



From strain gage measurements of strains ϵ_2 and ϵ_3 , we can find

$$E_2 = \frac{\sigma_2}{\epsilon_2} \quad \text{and} \quad \nu_{23} = -\frac{\epsilon_3}{\epsilon_2} = \nu_{32}$$

then, for the transversely isotropic material,

$$G_{23} = \frac{E_2}{2(1+\nu_{32})}$$

2.9

the off-axis modulus of elasticity is

$$E_x = \frac{\sigma_x}{\epsilon_x} = \frac{\sigma_x}{\bar{S}_{11} \sigma_x} = \frac{1}{\bar{S}_{11}} \quad \text{where } \sigma_y = \tau_{xy} = 0$$

$$\begin{aligned} \text{Where } \bar{S}_{11} &= S_{11} \cos^4 \theta + S_{22} \sin^4 \theta + (2S_{12} + S_{66}) \sin^2 \theta \cos^2 \theta \\ &= \frac{1}{E_1} \cos^4 \theta + \frac{1}{E_2} \sin^4 \theta + \left(-\frac{2\nu_{12}}{E_1} + \frac{1}{G_{12}} \right) \sin^2 \theta \cos^2 \theta \end{aligned}$$

Thus,

$$E_x = \frac{1}{\frac{1}{E_1} \cos^4 \theta + \frac{1}{E_2} \sin^4 \theta + \left(-\frac{2\nu_{12}}{E_1} + \frac{1}{G_{12}} \right) \sin^2 \theta \cos^2 \theta}$$

2.10

The off-axis shear modulus is

$$G_{xy} = \frac{\tau_{xy}}{\gamma_{xy}} = \frac{\tau_{xy}}{\bar{S}_{66} \tau_{xy}} = \frac{1}{\bar{S}_{66}} \quad \text{where } \sigma_x = \sigma_y = 0$$

$$\begin{aligned} \text{and where } \bar{S}_{66} &= 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66}) \sin^2 \theta \cos^2 \theta + S_{66} (\sin^4 \theta + \cos^4 \theta) \\ &= 2\left(\frac{2}{E_1} + \frac{2}{E_2} + \frac{4\nu_{12}}{E_1} - \frac{1}{G_{12}}\right) \sin^2 \theta \cos^2 \theta + \frac{1}{G_{12}} (\sin^4 \theta + \cos^4 \theta) \end{aligned}$$

thus,

$$G_{xy} = \frac{1}{2\left(\frac{2}{E_1} + \frac{2}{E_2} + \frac{4\nu_{12}}{E_1} - \frac{1}{G_{12}}\right) \sin^2 \theta \cos^2 \theta + \frac{1}{G_{12}} (\sin^4 \theta + \cos^4 \theta)}$$

2.11

Rearranging the result from Problem 2.10, we have

$$\frac{1}{G_{xy}} = A \sin^2 \theta \cos^2 \theta + B (\sin^4 \theta + \cos^4 \theta)$$

$$\text{where } A = 2\left(\frac{2}{E_1} + \frac{2}{E_2} + \frac{4\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)$$

$$\text{and } B = \frac{1}{G_{12}}$$

(a) For a possible maximum, minimum or inflection point, we must have

$$\frac{d}{d\theta} \left(\frac{1}{G_{xy}} \right) = (2A - 4B) (\sin \theta \cos \theta) (\cos^2 \theta - \sin^2 \theta) = 0$$

The three solutions for the angle θ are therefore

1. $\sin \theta = 0$, or $\theta = 0^\circ$
2. $\cos \theta = 0$, or $\theta = 90^\circ$
3. $\cos^2 \theta - \sin^2 \theta = 0$, or $\theta = 45^\circ$

(b) In order to find the bounds on G_{12} for a minimum or maximum of G_{xy} , we consider the sign of the derivative $\frac{d^2}{d\theta^2} \left(\frac{1}{G_{xy}} \right)$ at $\theta = 45^\circ$.

continued

2.11 continued

$$\frac{d^2}{d\theta^2} \left(\frac{1}{G_{xy}} \right) = \frac{d}{d\theta} \left[(2A-4B)(\sin\theta \cos\theta)(\cos^2\theta - \sin^2\theta) \right]$$

$$= (2A-4B)(\cos^4\theta + \sin^4\theta - 6\sin^2\theta \cos^2\theta)$$

and

$$\frac{d^2}{d\theta^2} \left(\frac{1}{G_{xy}} \right) = (2A-4B)(-1) = 4B-2A$$

$\theta = 45^\circ$

For $\frac{d^2}{d\theta^2} \left(\frac{1}{G_{xy}} \right) > 0$, $4B-2A > 0$, or

$\theta = 45^\circ$

$$G_{12} < \frac{1}{\frac{1}{E_1} + \frac{1}{E_2} + \frac{2\nu_{12}}{E_1}} \quad \text{for a minimum}$$

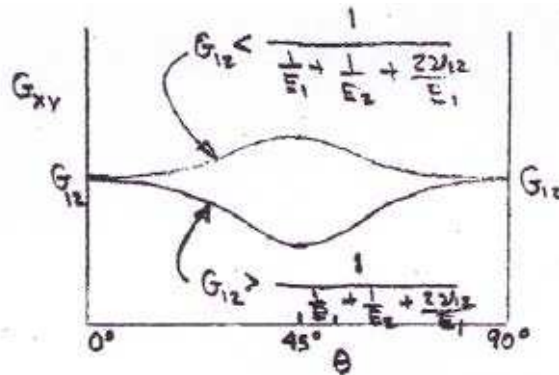
of $\frac{1}{G_{xy}}$ (or a maximum of G_{xy}).

For $\frac{d^2}{d\theta^2} \left(\frac{1}{G_{xy}} \right) < 0$, $4B-2A < 0$, or

$\theta = 45^\circ$

$$G_{12} > \frac{1}{\frac{1}{E_1} + \frac{1}{E_2} + \frac{2\nu_{12}}{E_1}} \quad \text{for a minimum of } G_{xy}$$

(c)



(d) From Table 2.2, for E-glass/epoxy (Scotchply 1002),
 $E_1 = 38.6 \text{ GPa}$, $E_2 = 8.27 \text{ GPa}$, $G_{12} = 4.14 \text{ GPa}$, $\nu_{12} = 0.26$

continued

2.11 continued

Therefore

$$\frac{1}{\frac{1}{E_1} + \frac{1}{E_2} + \frac{2\nu_{12}}{E_1}} = \frac{1}{\frac{1}{38.6} + \frac{1}{8.77} + \frac{2(0.26)}{38.6}} = 6.24 \text{ GPa}$$

and since $6.24 \text{ GPa} > (G_{12} = 4.14 \text{ GPa})$, the value of G_{xy} at $\theta = 45^\circ$ must be a maximum for this material, as shown by the upper curve in part (c).

2.12

From longitudinal tension test with $\sigma_1 \neq 0$, $\sigma_2 = \tau_{12} = 0$, we can measure longitudinal strains, ϵ_1 , and calculate the longitudinal modulus $E_1 = \sigma_1 / \epsilon_1$. From measured transverse strains, ϵ_2 , we can find the major Poisson's ratio $\nu_{12} = -\epsilon_2 / \epsilon_1$. Similarly, from transverse tension test with $\sigma_2 \neq 0$, $\sigma_1 = \tau_{12} = 0$, we can find the transverse modulus $E_2 = \sigma_2 / \epsilon_2$. Finally, from an off-axis tension test at a known angle, θ , we can determine the off-axis modulus $E_x = \sigma_x / \epsilon_x$. Then from Eq. (2.40), we can solve for G_{12} , since E_x, E_1, E_2, ν_{12} and θ are known,

2.13

First, we transform the stresses to the 1, 2 system

$$\begin{aligned} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} &= \begin{bmatrix} \cos^2 \theta & \sin^2 \theta & 2 \cos \theta \sin \theta \\ \sin^2 \theta & \cos^2 \theta & -2 \cos \theta \sin \theta \\ -\cos \theta \sin \theta & \cos \theta \sin \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \\ &= \begin{bmatrix} 0.75 & 0.25 & 0.866 \\ 0.25 & 0.75 & -0.866 \\ -0.433 & 0.433 & 0.5 \end{bmatrix} \begin{Bmatrix} 100 \\ -50 \\ 50 \end{Bmatrix} = \begin{Bmatrix} 105.8 \\ -55.8 \\ -39.95 \end{Bmatrix} \text{ MPa} \end{aligned}$$

the strains in the 1, 2 system are then

continued

2.13 continued

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_1} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

2.14

Eqs. (2.37) are as follows:

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

the $\bar{S}_{ij}$ can be expressed in terms of off-axis engineering constants by considering the strains that are generated by simple uniaxial tension and shear experiments.

For example, a uniaxial tension test with $\sigma_x \neq 0$, $\sigma_y = \tau_{xy} = 0$ will generate the strains

$$\epsilon_x = \frac{\sigma_x}{E_x} = \bar{S}_{11} \sigma_x$$

$$\epsilon_y = -\nu_{xy} \epsilon_x = -\nu_{xy} \frac{\sigma_x}{E_x} = \bar{S}_{12} \sigma_x$$

$$\gamma_{xy} = \eta_{x,xy} \epsilon_x = \eta_{x,xy} \frac{\sigma_x}{E_x} = \bar{S}_{16} \sigma_x$$

continued

2.14 continued

Similarly, a uniaxial tension test with $\sigma_y \neq 0$, $\sigma_x = \tau_{xy} = 0$ will generate the strains

$$\epsilon_y = \frac{\sigma_y}{E_y} = \bar{S}_{22} \sigma_y$$

$$\epsilon_x = -\nu_{yx} \epsilon_y = -\nu_{yx} \frac{\sigma_y}{E_y} = \bar{S}_{12} \sigma_y$$

$$\gamma_{xy} = \eta_{yx,xy} \epsilon_y = \eta_{yx,xy} \frac{\sigma_y}{E_y} = \bar{S}_{26} \sigma_y$$

Finally, a pure shear test with $\tau_{xy} \neq 0$, $\sigma_x = \sigma_y = 0$ will generate the strains

$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} = \bar{S}_{16} \tau_{xy}$$

$$\epsilon_x = \eta_{xy,x} \gamma_{xy} = \eta_{xy,x} \frac{\tau_{xy}}{G_{xy}} = \bar{S}_{16} \tau_{xy}$$

$$\epsilon_y = \eta_{xy,y} \gamma_{xy} = \eta_{xy,y} \frac{\tau_{xy}}{G_{xy}} = \bar{S}_{26} \tau_{xy}$$

Using superposition of the strains for a general state of stress σ_x , σ_y , τ_{xy} , we find that

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & \frac{\eta_{xy,x}}{G_{xy}} \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{\eta_{xy,y}}{G_{xy}} \\ \frac{\eta_{yx,x}}{E_x} & \frac{\eta_{yx,y}}{E_y} & \frac{1}{G_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

or, due to the symmetry of the $\bar{S}_{ij}$ matrix,

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & \frac{\eta_{xy,x}}{G_{xy}} \\ -\frac{\nu_{yx}}{E_y} & \frac{1}{E_y} & \frac{\eta_{xy,y}}{G_{xy}} \\ \frac{\eta_{xy,x}}{G_{xy}} & \frac{\eta_{xy,y}}{G_{xy}} & \frac{1}{G_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

2.15

from Eqs.(2.36):

$$\bar{Q}_{11} = Q_{11} \cos^4 \theta + Q_{22} \sin^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta$$

Substituting in the following trigonometric identities:

$$\cos^4 \theta = \frac{1}{8} (3 + 4 \cos 2\theta + \cos 4\theta)$$

$$\sin^4 \theta = \frac{1}{8} (3 - 4 \cos 2\theta + \cos 4\theta)$$

$$\cos^2 \theta \sin^2 \theta = \frac{1}{8} (1 - \cos 4\theta)$$

we find that

$$\begin{aligned} \bar{Q}_{11} &= \frac{1}{8} (3 + 4 \cos 2\theta + \cos 4\theta) Q_{11} \\ &\quad + \frac{1}{8} (3 - 4 \cos 2\theta + \cos 4\theta) Q_{22} \\ &\quad + \frac{1}{4} (1 - \cos 4\theta) (Q_{12} + 2Q_{66}) \\ &= U_1 + U_2 \cos 2\theta + U_3 \cos 4\theta \end{aligned}$$

where

$$U_1 = \frac{1}{8} (3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66})$$

$$U_2 = \frac{1}{2} (Q_{11} - Q_{22})$$

$$U_3 = \frac{1}{8} (Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66})$$

Similarly,

$$\begin{aligned} \bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66}) \sin^2 \theta \cos^2 \theta \\ &\quad + Q_{12} (\cos^4 \theta + \sin^4 \theta) \\ &= U_1 - U_3 \cos 4\theta \end{aligned}$$

where

$$U_4 = \frac{1}{8} (Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66})$$

2.16

From Eqs. (2.45):

$$U_1 = \frac{1}{8} [3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66}]$$

$$= \frac{1}{8} [(3)(138.8) + (3)(9.05) + (2)(2.716) + (4)(6.9)] = 59.57 \text{ GPa}$$

$$U_2 = \frac{1}{2} [Q_{11} - Q_{22}] = \frac{1}{2} [138.8 - 9.05] = 64.87 \text{ GPa}$$

$$U_3 = \frac{1}{8} [Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66}]$$

$$= \frac{1}{8} [138.8 + 9.05 - (2)(2.716) - (4)(6.9)] = 14.35 \text{ GPa}$$

$$U_4 = \frac{1}{8} [Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66}]$$

$$= \frac{1}{8} [138.8 + 9.05 + (6)(2.716) - (4)(6.9)] = 17.07 \text{ GPa}$$

From Eqs. (2.47):

$$V_1 = \frac{1}{8} [3S_{11} + 3S_{22} + 2S_{12} + S_{66}]$$

$$= \frac{1}{8} [(3)(0.00725) + (3)(0.111) + (2)(-0.00217) + 0.145]$$

$$= 0.0619 \text{ (GPa)}^{-1}$$

$$V_2 = \frac{1}{2} [S_{11} - S_{12}] = \frac{1}{2} [0.00725 - (-0.00217)] = 0.0047 \text{ (GPa)}^{-1}$$

$$V_3 = \frac{1}{8} [S_{11} + S_{22} - 2S_{12} - S_{66}]$$

$$= \frac{1}{8} [0.00725 + 0.111 - (2)(-0.00217) - 0.145]$$

$$= -0.0028 \text{ (GPa)}^{-1}$$

$$V_4 = \frac{1}{8} [S_{11} + S_{22} + 6S_{12} - S_{66}]$$

$$= \frac{1}{8} [0.00725 + 0.111 + (6)(-0.00217) - 0.145]$$

$$= -0.00497 \text{ (GPa)}^{-1}$$

2.17

From Eqs. (2.44), for $\theta = +45^\circ$, we have

$$\begin{aligned}\bar{Q}_{11} &= U_1 + U_2 \cos 2\theta + U_3 \cos 4\theta \\ &= 59.57 + 64.87 \cos 90^\circ + 14.35 \cos 180^\circ = 45.22 \text{ GPa}\end{aligned}$$

$$\bar{Q}_{12} = U_4 - U_3 \cos 4\theta = 17.07 - 14.35 \cos 180^\circ = 31.42 \text{ GPa}$$

$$\begin{aligned}\bar{Q}_{22} &= U_1 - U_2 \cos 2\theta + U_3 \cos 4\theta \\ &= 59.57 - 64.87 \cos 90^\circ + 14.35 \cos 180^\circ = 45.22 \text{ GPa}\end{aligned}$$

$$\begin{aligned}\bar{Q}_{16} &= \frac{U_2}{2} \sin 2\theta + U_3 \sin 4\theta \\ &= \frac{64.87}{2} \sin 90^\circ + 14.35 \sin 180^\circ = 32.44 \text{ GPa}\end{aligned}$$

$$\begin{aligned}\bar{Q}_{26} &= \frac{U_2}{2} \sin 2\theta - U_3 \sin 4\theta \\ &= \frac{64.87}{2} \sin 90^\circ - 14.35 \sin 180^\circ = 32.44 \text{ GPa}\end{aligned}$$

$$\begin{aligned}\bar{Q}_{66} &= \frac{1}{2} (U_1 - U_4) - U_3 \cos 4\theta \\ &= \frac{1}{2} (59.57 - 17.07) - 14.35 \cos 180^\circ = 35.6 \text{ GPa}\end{aligned}$$

The same results can be obtained by using Eqs. (2.36).

For $\theta = -45^\circ$, using either Eqs. (2.44) or Eqs. (2.36), we find that $\bar{Q}_{11}$, $\bar{Q}_{12}$, $\bar{Q}_{22}$ and $\bar{Q}_{66}$ are the same as for $\theta = +45^\circ$. The values of $\bar{Q}_{16}$ and $\bar{Q}_{26}$ are just the negatives of the corresponding values at $\theta = +45^\circ$, however. Thus, for $\theta = -45^\circ$, we get

$$\bar{Q}_{11} = 45.22 \text{ GPa}$$

$$\bar{Q}_{12} = 31.42 \text{ GPa}$$

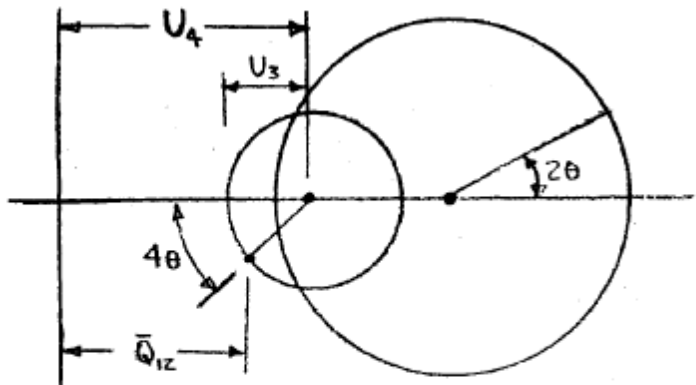
$$\bar{Q}_{22} = 45.22 \text{ GPa}$$

$$\bar{Q}_{16} = -32.44 \text{ GPa}$$

$$\bar{Q}_{26} = -32.44 \text{ GPa}$$

$$\bar{Q}_{66} = 35.6 \text{ GPa}$$

2.18



2.19

$$\bar{Q}_{12} = U_4 - U_3 \cos 4\theta$$

The averaged stiffnesses are:

$$\tilde{Q}_{11} = \frac{\int_0^\pi \bar{Q}_{11} d\theta}{\int_0^\pi d\theta} = \frac{\int_0^\pi (U_1 + U_2 \cos 2\theta + U_3 \cos 4\theta) d\theta}{\pi} = U_1$$

Similarly,

$$\tilde{Q}_{22} = \frac{\int_0^\pi \bar{Q}_{22} d\theta}{\pi} = \frac{\int_0^\pi (U_1 - U_2 \cos 2\theta + U_3 \cos 4\theta) d\theta}{\pi} = U_1$$

$$\tilde{Q}_{12} = \frac{\int_0^\pi \bar{Q}_{12} d\theta}{\pi} = \frac{\int_0^\pi (U_4 - U_3 \cos 4\theta) d\theta}{\pi} = U_4 = \tilde{Q}_{21}$$

$$\tilde{Q}_{66} = \frac{\int_0^\pi \bar{Q}_{66} d\theta}{\pi} = \frac{\int_0^\pi \left[\frac{1}{2} (U_1 - U_4) - U_3 \cos 4\theta \right] d\theta}{\pi} = \frac{U_1 - U_4}{2}$$

$$\tilde{Q}_{16} = \tilde{Q}_{26} = 0$$

The stress-strain relationships are therefore

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} & 0 \\ \tilde{Q}_{12} & \tilde{Q}_{22} & 0 \\ 0 & 0 & \tilde{Q}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} U_1 & U_4 & 0 \\ U_4 & U_1 & 0 \\ 0 & 0 & \frac{U_1 - U_4}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

continued

2.19 continued

Since the material is planar isotropic, we have

$$\tilde{Q}_{11} = U_1 = \frac{\tilde{E}}{1 - \tilde{\nu}^2} = \tilde{Q}_{22}$$

$$\tilde{Q}_{12} = U_4 = \frac{\tilde{\nu} \tilde{E}}{1 - \tilde{\nu}^2}$$

$$\tilde{Q}_{66} = \frac{U_1 - U_4}{2} = \tilde{G} = \frac{\tilde{E}}{2(1 + \tilde{\nu})}$$

Where $\tilde{E}$ = averaged modulus of elasticity
 $\tilde{G}$ = averaged shear modulus
 $\tilde{\nu}$ = averaged Poisson's ratio

Solving these equations simultaneously, we get

$$\tilde{E} = \frac{(U_1 - U_4)(U_1 + U_4)}{U_1}$$

$$\tilde{G} = \frac{U_1 - U_4}{2}$$

$$\tilde{\nu} = \frac{U_4}{U_1}$$

2.20

$$E_x = \frac{\sigma_x}{\epsilon_x} = \frac{100 \text{ MPa}}{0.00647} = 15,456 \text{ MPa} = 15,456 \text{ GPa}$$

$$\nu_{xy} = - \frac{\epsilon_y}{\epsilon_x} = - \frac{(-0.00324)}{0.00647} = 0.5$$

$$\eta_{x,xy} = \frac{\gamma_{xy}}{\epsilon_x}$$

Where the shear strain γ_{xy} is found from the measured strains ϵ_x, ϵ_y and ϵ_1 and the strain transformation equation (Eq. 2.33) evaluated at $\theta = 45^\circ$.

$$\epsilon_1 = \epsilon_x \cos^2 \theta + \epsilon_y \sin^2 \theta + \left(\frac{\gamma_{xy}}{2} \right) (2 \cos \theta \sin \theta)$$

or

$$\gamma_{xy} = \frac{\epsilon_1 - \epsilon_x \cos^2 \theta - \epsilon_y \sin^2 \theta}{\cos \theta \sin \theta}$$

$$\gamma_{xy} = \frac{0.00809 - 0.00647(0.5) - (-0.00324)(0.5)}{0.5}$$

$$\gamma_{xy} = 0.01295$$

so that

$$\eta_{x,xy} = \frac{\gamma_{xy}}{\epsilon_x} = \frac{0.01295}{0.00647} = 2.0$$

2.21

From the first of Eqs (2.40):

$$\frac{1}{E_x} = \frac{c^4}{E_1} + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1} \right) s^2 c^2 + \frac{1}{E_2} s^4$$

From Table 2.2 for AS/3501:

$$E_1 = 138 \text{ GPa}, E_2 = 9 \text{ GPa}, \nu_{12} = 0.3, G_{12} = 6.9 \text{ GPa}$$

$$\therefore \frac{1}{E_x} = \frac{(0.707)^4}{138} + \left(\frac{1}{6.9} - \frac{2(0.3)}{138} \right) (0.707)^4 + \frac{(0.707)^4}{9}$$

$$E_x = 15.447 \text{ GPa}$$

So that

$$E_x = \frac{\sigma_x}{\epsilon_x} = \frac{15.44 \text{ MPa}}{\epsilon_x} = 15.447 \text{ GPa}$$

$$\therefore \epsilon_x = \frac{\sigma_x}{E_x} = \frac{15.44 \text{ MPa}}{15.447 \text{ GPa}} = 0.001$$

$$\begin{aligned} 2.22 (a) \quad \tau_{xy} &= \sigma_1 c s - \sigma_2 c s + \tau_{12} (c^2 - s^2) \\ &= 1000(0.707)^2 - (-1000)(0.707)^2 + 0 \\ &= 1000 \text{ MPa} \end{aligned}$$

$$\begin{aligned} (b) \quad \epsilon_1 &= \frac{\sigma_1}{E_1} - \frac{\nu_{12}}{E_1} \sigma_2 \\ &= \frac{1.0}{131} - 0.22 \left(\frac{-1.0}{131} \right) \\ &= 0.0093 \end{aligned}$$

$$\begin{aligned} \epsilon_2 &= -\frac{\nu_{12}}{E_1} \sigma_1 + \frac{\sigma_2}{E_2} \\ &= -\frac{0.22}{131} + \frac{(-1)}{10.3} \\ &= -0.0988 \end{aligned}$$

$$\gamma_{12} = \frac{\tau_{12}}{G_{12}} = 0$$

2.23

In terms of invariants, $\bar{Q}_{66} = \frac{U_1 - U_4}{2} - U_3 \cos 4\theta$

For a maximum of $\bar{Q}_{66}$, we must have

$$\frac{d\bar{Q}_{66}}{d\theta} = 4U_3 \sin 4\theta = 0 \text{ and } \frac{d^2\bar{Q}_{66}}{d\theta^2} = 16U_3 \cos 4\theta <$$

the possible solutions are $\theta = 0, \frac{\pi}{4}$ and $\frac{\pi}{2}$

at $\theta = 0^\circ$, $\frac{d^2\bar{Q}_{66}}{d\theta^2} = 16U_3 \cos(0) = 16U_3 > 0$ (since $U_3 >$

at $\theta = \frac{\pi}{4}$, $\frac{d^2\bar{Q}_{66}}{d\theta^2} = 16U_3 \cos(\pi) = -16U_3 < 0$

at $\theta = \frac{\pi}{2}$, $\frac{d^2\bar{Q}_{66}}{d\theta^2} = 16U_3 \cos(2\pi) = 16U_3 > 0$

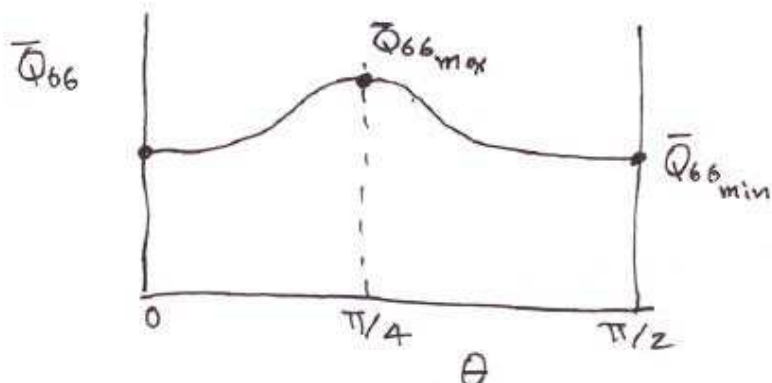
$\therefore \bar{Q}_{66_{\max}}$ occurs at $\theta = \frac{\pi}{4}$ and

$$\bar{Q}_{66_{\max}} = \frac{U_1 - U_4}{2} - U_3 \cos(\pi) = \frac{U_1 - U_4}{2} + U_3$$

while $\bar{Q}_{66_{\min}}$ occurs at $\theta = 0$ and $\frac{\pi}{2}$, where

$$\bar{Q}_{66_{\min}} = \frac{U_1 - U_4}{2} - U_3 \cos(0) = \frac{U_1 - U_4}{2} - U_3$$

Graphically, the variation of $\bar{Q}_{66}$ with θ is as shown below



Chapter 3

3.1

Consider the rectangular area in Fig. 3.5, the total composite area is $A_c = ab$, while the area of fiber enclosed in the total area is

$$A_f = \pi \left(\frac{s_a}{2} \right) \left(\frac{s_b}{2} \right)$$

the fiber area fraction, which is the same as the fiber volume fraction, is then

$$V_f = \frac{A_f}{A_c} = \frac{\pi s_a s_b}{4ab}$$

the maximum fiber volume fraction is achieved when

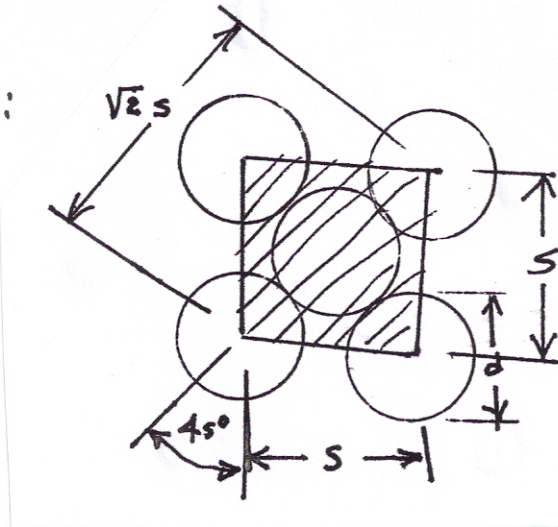
$$s_a = a \text{ and } s_b = b$$

or

$$V_{fmax} = \frac{\pi}{4} = 0.785$$

3.2

FCC array:



For the cross-hatched area above,

$$V_f = \frac{(2)(\pi d^2/4)}{s^2} = \frac{\pi d^2}{2s^2}$$

When the fibers contact, $\sqrt{2}s = 2d$ and

$$V_{fmax} = \frac{\pi d^2}{4d^2} = \frac{\pi}{4} = 0.785$$

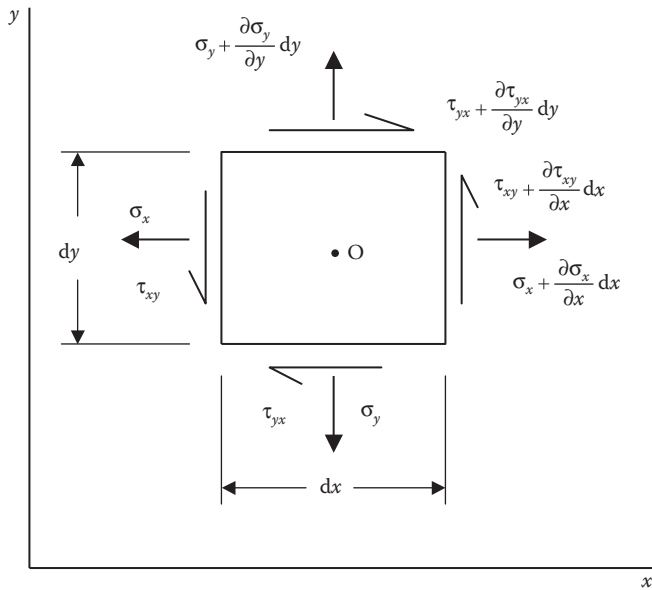


FIGURE B.1

Infinitesimal element representing stresses at a point in a body.