

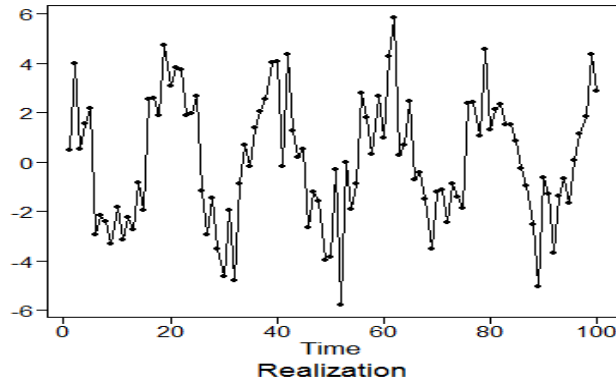
***Applied Time Series Analysis with R 2nd edition
Woodward Solutions Manual***

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CHAPTER 2 – Problem Solutions

Problem 2.1 Below is the plot a data set generated using

```
x=gen.sigplusnoise.wge(100,b0=0,b1=0,coef=c(3,1.5),freq=c(.05,.35),psi=c(0,2))
```



(a) If x is the data set generated above, then the following command produces the low-pass filtered data.

```
xl2=butterworth.wge(x,order=3,type='low',cutoff=.2)
```

(b) If x is the data set generated above, then the following command produces the high-pass filtered data.

```
xh2=butterworth.wge(x,order=3,type='high',cutoff=.2)
```

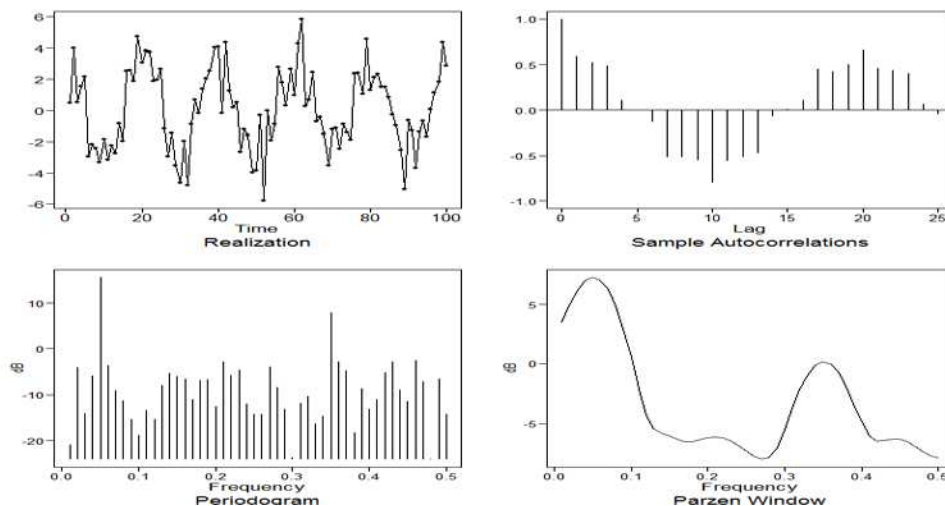
(c) If $xh2$ is the high-pass filtered data in (b) then use the following command:

```
xhl2=butterworth.wge(xh2$x.filt,order=3,type='low',cutoff=.2)
```

```
xl2=butterworth.wge(x,oder=3,type='high',cutoff=.2)
```

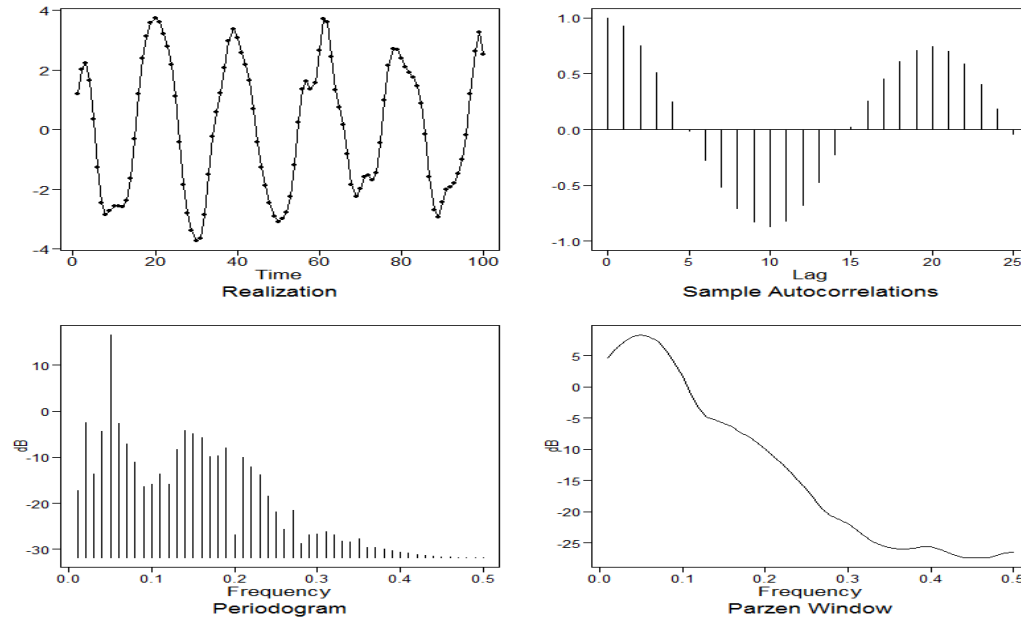
Original data set: command `plotts.sample.wge(x)`

This signal is characterized by cyclic behavior (with period about 20) along with a high-frequency component. The sample autocorrelations primarily show the cyclic behavior with period about 20 but may be affected slightly by the high-frequency behavior. The Parzen window (and periodogram) show two peaks at about .05 and .35.



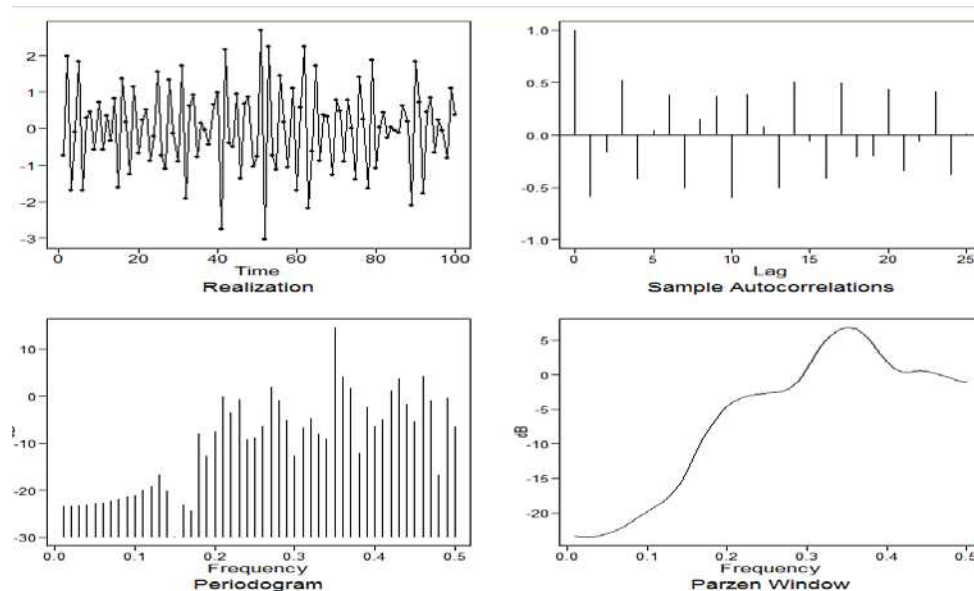
Low-pass filtered data set (`x12`): `command plotts.sample.wge(x12)`

This signal is characterized by a smooth cyclic behavior (with period about 20) with the high-frequency component removed. The sample autocorrelations clearly show the cyclic behavior with period about 20. The Parzen window (and periodogram) show a peak at about .05.



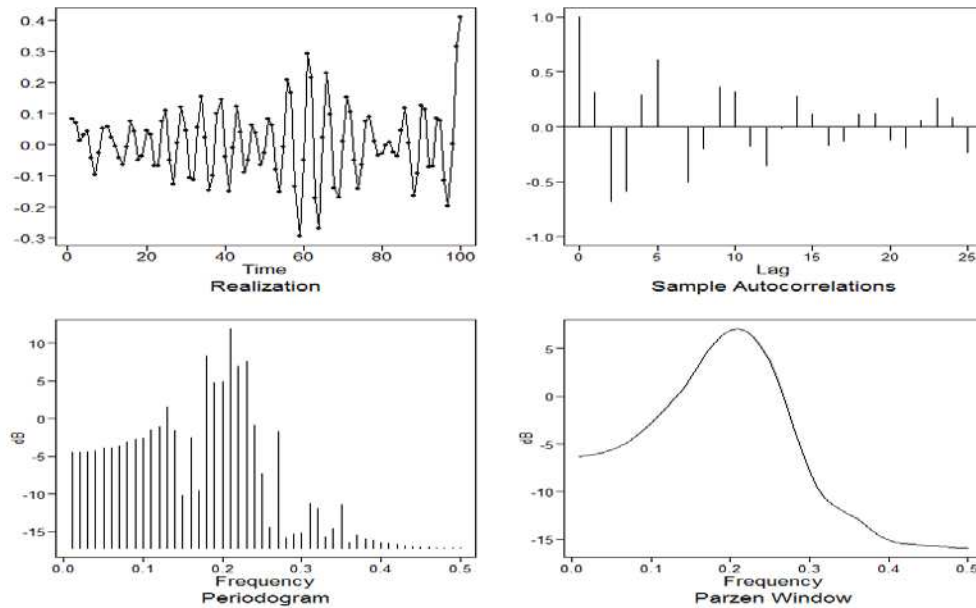
High-pass filtered data set (`xh2$x.filt`): `command xh2=plotts.sample.wge(x)`

This signal is characterized by a high-frequency (nearly up-and-down) behavior. The sample autocorrelations clearly show the cyclic behavior with period about 3. The Parzen window (and periodogram) show a peak at about .35.



Low-pass filtering the high-pass filtered data: `xl12=plotts.sample.wge(xh2$x.filt)`

This signal is very weak. Whereas the original signal went from -6 to 6, the low-pass data went from -4 to 4 and the high-pass data had range -2 to 2, the double filtered data set goes from about -0.4 to 0.4. It would be clearer if all plots were plotted on the same scale. While we might have thought the double-filtered data would be essentially white noise, there does seem to be some periodic behavior with period about 4-5 as characterized by the data, sample autocorrelations and spectrum.



Problem 2.2 (a) If x is the data set generated in Problem 2.1, then the following command produces the low-pass filtered data.

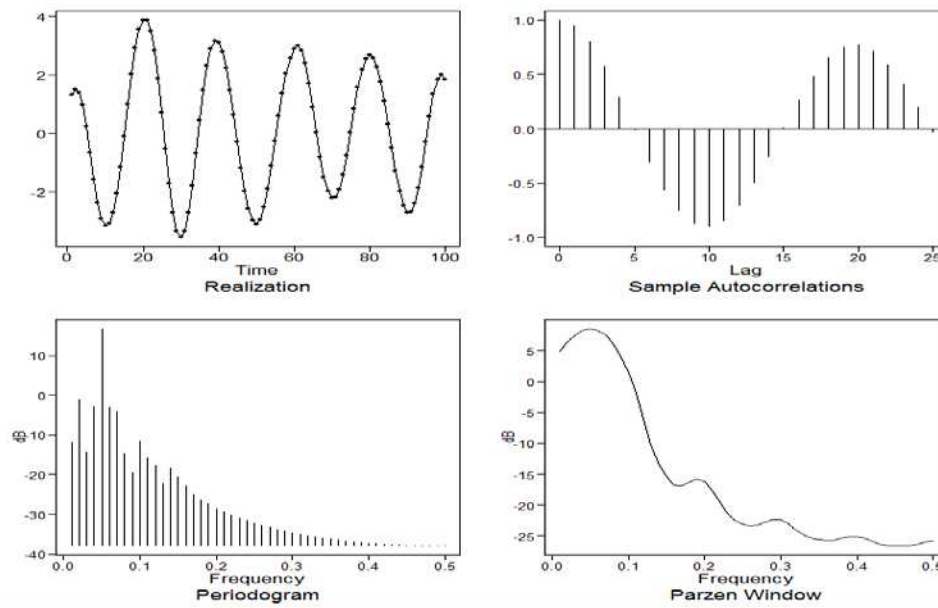
```
xl1=butterworth.wge(x,order=4,type='low',cutoff=.1)
```

(b) If x is the data set generated above, then the following command produces the high-pass filtered data.

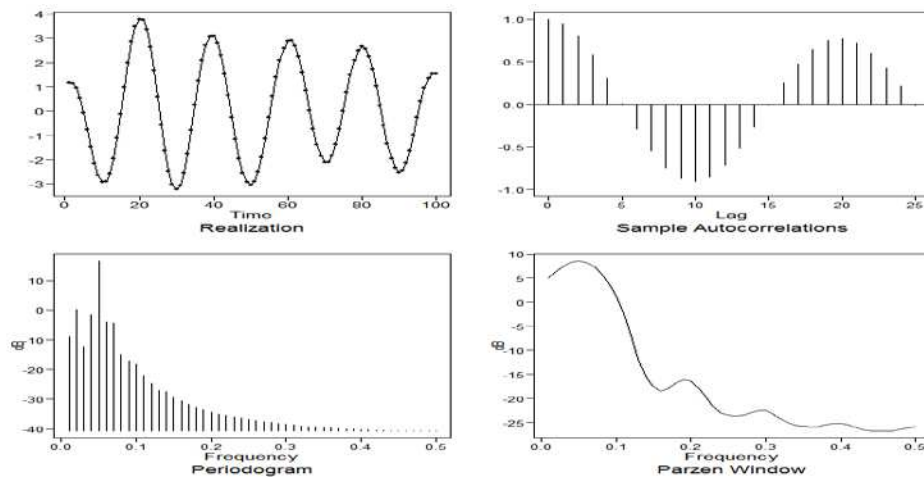
```
xl075=butterworth.wge(x,order=4,type='high',cutoff=.075)
```

The two sets of plots below are very similar to each other. Comparing the filtered data with the original data for the low-pass filtered data with `cutoff=.2`, we see that the data with cutoffs closer to 0.05 are smoother and show no impact of the high-frequency behavior.

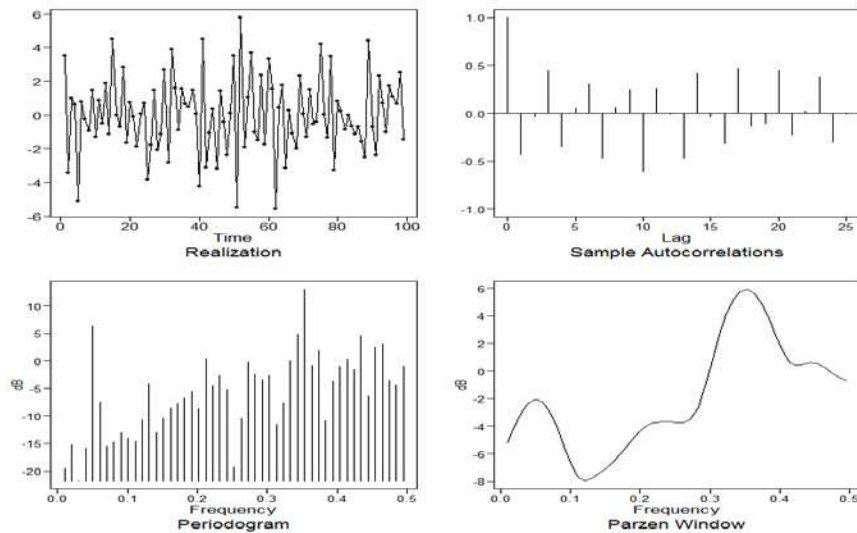
(a) `command` `plots.sample.wge(xl1$x.filt)`



(b) `command` `plots.sample.wge(xl075$x.filt)`



Problem 2.3 command `xdif=artrans,wge(x,phi.dif=1)`
`plots.sample.wge(xdif)`



The filtering weakened the frequency component at $f=0.05$ and served as a high-pass filter. As Figure 2.2b shows, the filter allows some frequency behavior as low as 0.05 to leak into the filtered data. It is not as good as the Butterworth filter for filtering out one of the two signals.

Problem 2.4

2-point:

$$h_0 = \frac{1}{2}, h_{-1} = \frac{1}{2}, h = 0, \text{elsewhere}$$

$$\begin{aligned} H(e^{-2\pi if}) &= \sum_{j=0}^1 h_j e^{-2\pi ifj} = 1 + 1 \times e^{-2\pi if} \\ &= \frac{1}{2} + \frac{1}{2}(\cos(2\pi f) - i \sin(2\pi f)) \\ &= \end{aligned}$$

$$\begin{aligned} |H(e^{-2\pi if})|^2 &= \left(\frac{1}{2} + \frac{1}{2}(\cos(2\pi f) - i \sin(2\pi f))\right)^2 \\ &= \frac{1}{4} + \frac{1}{2}\cos(2\pi f) + \frac{1}{4}\cos^2(2\pi f) + \frac{1}{4}\sin^2(2\pi f) \\ &= \frac{1}{2}(1 + \cos(2\pi f)) \end{aligned}$$

3-point

$$h_0 = \frac{1}{3}, h_{-1} = \frac{1}{3}, h_1 = \frac{1}{3}, h = 0, \text{elsewhere}$$

$$\begin{aligned} H(e^{-2\pi if}) &= \sum_{j=-1}^1 h_j e^{-2\pi ifj} \\ &= \frac{1}{3}(\cos(2\pi f + i \sin(2\pi f)) + \frac{1}{3} + \frac{1}{3}(\cos(2\pi f) - i \sin(2\pi f))) \\ &= \frac{1}{3}(1 + 2\cos(2\pi f)) \end{aligned}$$

$$|H(e^{-2\pi if})|^2 = \frac{1}{9}(1 + 2\cos(2\pi f))^2$$

5-point

$$h_0 = h_{-1} = h_{-2} = h_1 = h_2 = 1/5, h_j = 0, \text{elsewhere}$$

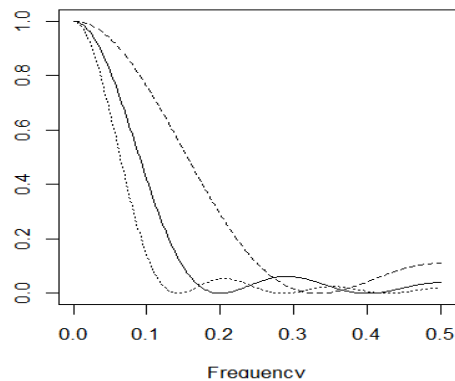
$$\begin{aligned} H(e^{-2\pi if}) &= \sum_{j=-2}^2 h_j e^{-2\pi ifj} \\ &= 1/5 (\cos(4\pi f) + i\sin(4\pi f)) + \cos(2\pi f) + i\sin(2\pi f) + 1 + (\cos(2\pi f) - i\sin(2\pi f) + \cos(4\pi f) - i\sin(4\pi f)) \\ &= 1/5 (1 + 2\cos(2\pi f) + 2\cos(4\pi f)) \\ |H(e^{-2\pi if})|^2 &= 1/9 (1 + 2\cos(2\pi f) + 2\cos(4\pi f))^2 \end{aligned}$$

7-point

$$h_0 = h_{-1} = h_{-2} = h_{-3} = h_1 = h_2 = h_3 = 1/7, h_j = 0, \text{elsewhere}$$

$$\begin{aligned} H(e^{-2\pi if}) &= \sum_{j=-3}^3 h_j e^{-2\pi ifj} = \\ &= 1/7 (\cos(6\pi f) + i\sin(6\pi f) + \cos(4\pi f) + i\sin(4\pi f)) + \cos(2\pi f) + i\sin(2\pi f) + 1 + (\cos(2\pi f) - i\sin(2\pi f) + \cos(4\pi f) - i\sin(4\pi f) \\ &\quad + \cos(6\pi f) - i\sin(6\pi f)) \\ &= 1/7 (1 + 2\cos(2\pi f) + 2\cos(4\pi f) + 2\cos(6\pi f)) \\ |H(e^{-2\pi if})|^2 &= 1/49 (1 + 2\cos(2\pi f) + 2\cos(4\pi f) + 2\cos(6\pi f))^2 \end{aligned}$$

The desired plot of the squared frequency response for a 5-point moving average (solid line). Also shown are the 3-point (dashed line) and 7-point (dotted line). The R code to create these plots is also shown.



```
ff=1:251
f=(ff-1)/500
H5=(1+2*cos(2*pi*f)+2*cos(4*pi*f))^2/25
plot(f,H5,type='l',xlab='Frequency')
H3=(1+2*cos(2*pi*f))^2/9
lines(f,H3,type='l',lty=2)
H7=(1+2*cos(6*pi*f)+2*cos(4*pi*f)+2*cos(2*pi*f))^2/49
lines(f,H7,type='l',lty=3)
```

Problem 2.5

By Minkowski inequality

$$\begin{aligned} E[((a_1 X_t^{(m)} + b_1 Y_t^{(m)}) - (a_1 X_t + b_1 Y_t))^2] &\leq \left\{ [E(a_1 X_t^{(m)} - a_1 X_t)^2]^{1/2} + [E(b_1 Y_t^{(m)} - b_1 Y_t)^2]^{1/2} \right\}^2 \\ &= \left\{ a_1 [E(X_t^{(m)} - X_t)^2]^{1/2} + b_1 [E(Y_t^{(m)} - Y_t)^2]^{1/2} \right\}^2 \end{aligned}$$

By assumption both $E(X_t^{(m)} - X_t)^2 \rightarrow 0$ and $E(Y_t^{(m)} - Y_t)^2 \rightarrow 0$ as $m \rightarrow \infty$

So, $\lim_{m \rightarrow \infty} E[((a_1 X_t^{(m)} + b_1 Y_t^{(m)}) - (a_1 X_t + b_1 Y_t))^2] = 0$, i.e. $a_1 X_t^{(m)} + b_1 Y_t^{(m)} \rightarrow a_1 X_t + b_1 Y_t$ in mean square.

Problem 2.6

($\Rightarrow$) $Y = \sum_{t=1}^{\infty} X_t$ exists as a limit in mean square with $Y \in L^2$

Define $Y_t^{(m)} = \sum_{t=1}^m X_t$ $E[(Y_t^{(n)} - Y_t^{(m)})^2] \rightarrow 0$ as $m, n \rightarrow \infty$.

therefore, $E[\sum_{t=m+1}^n X_t]^2 \rightarrow 0$ as $m, n \rightarrow \infty$.

Since $\{X_t\}$ is a sequence of independent r.v.'s then $E(X_i X_j) = E(X_i)E(X_j) = 0, i \neq j$, b so

$$\begin{aligned} E(\sum_{t=m+1}^n X_t)^2 &= E(\sum_{t=m+1}^n X_t)^2 \text{ since the } X_t \text{'s are independent.} \\ &= \sum_{t=m+1}^n \sigma_t^2 \rightarrow 0 \text{ as } m, n \rightarrow \infty \text{ and consequently } \sum_{t=1}^{\infty} \sigma_t^2 < \infty. \end{aligned}$$

($\Leftarrow$) $\sum_{t=1}^{\infty} \sigma_t^2 < \infty$ from which it follows that $\sum_{t=m+1}^n \sigma_t^2 \rightarrow 0$ as $m, n \rightarrow \infty$. So, from above

$E[(Y_t^{(n)} - Y_t^{(m)})^2] = E[\sum_{t=m+1}^n X_t]^2 = \sum_{t=m+1}^n \sigma_t^2 \rightarrow 0$ as $m, n \rightarrow \infty$. Therefore, $Y_t = \sum_{t=1}^{\infty} X_t$ exists as a limit in mean square and is in L^2 .

Problem 2.7 We will show it for $k=2$ and the proof can be completed by induction.

$$X_t = b_1 X_t^1 + b_2 X_t^2 = (b_1 \sum_{j=1}^{\infty} h_j^{(1)} + b_2 \sum_{j=1}^{\infty} h_j^{(2)}) a_{t-j}$$

and since $(|b_1| \sum_{j=1}^{\infty} |h_j^{(1)}| + |b_2| \sum_{j=1}^{\infty} |h_j^{(2)}|) < \infty$, it follows from Theorem 2.3 that X_t is covariance stationary.

Problem 2.8 Since X_t is by definition a stationary process (see theorem 2.3)(a)

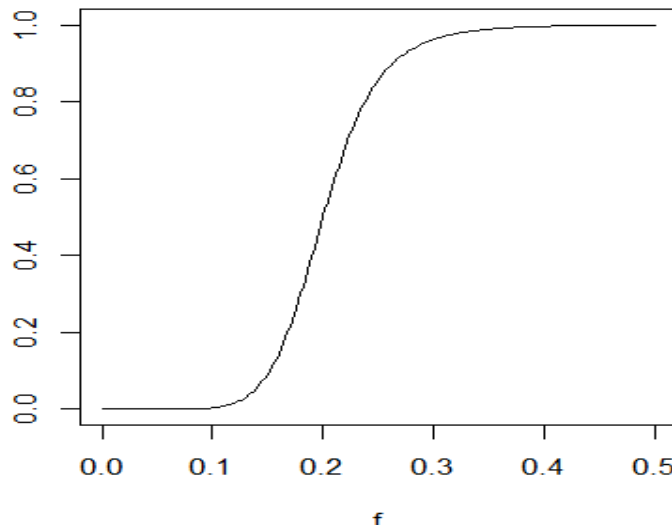
(a) Note that $\lim_{n \rightarrow \infty} \left| \sum_{k=1}^n h_k E(Z_{t-k}) \right| \leq \lim_{n \rightarrow \infty} \sum_{k=1}^n |h_k| \|E(Z_{t-k})\| < \infty$ so the limit exists

$$\begin{aligned}
 (b) E(X_t Y_t) &= E\left(\sum_{k=0}^{\infty} h_k Z_{t-k}\right) \left(\sum_{j=0}^{\infty} b_j Z_{t-j}\right) = \lim_{n \rightarrow \infty} E\left(\sum_{k=0}^n h_k Z_{t-k}\right) \left(\sum_{j=0}^n b_j Z_{t-j}\right) \\
 &= \lim_{n \rightarrow \infty} E\left(\sum_{k=0}^n \sum_{j=0}^n h_k b_j Z_{t-k} Z_{t-j}\right) \leq \lim_{n \rightarrow \infty} \sum_{k=0}^n \sum_{j=0}^n |h_k| |b_j| |E(Z_{t-k} Z_{t-j})| \\
 &= \lim_{n \rightarrow \infty} \sum_{k=0}^n \sum_{j=0}^n |h_k| |b_j| |E(Z_{t-k} Z_{t-j})| \leq \lim_{n \rightarrow \infty} \sum_{k=0}^n |h_k| \text{Var}(Z_{t-k}) \text{ since } |\gamma_h| \leq \gamma_0 \\
 &\leq \lim_{n \rightarrow \infty} \sum_{k=0}^n |h_k| M < \infty. \text{ Therefore the limit exists and } E(X_t Y_t) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} h_k b_j E(Z_{t-k} Z_{t-j}).
 \end{aligned}$$

Problem 2.9 I will examine the type of filter by plotting $|H(e^{-2\pi i f})|^2$.

(a) Using the code `ff=1:251`
`f=(ff-1)/500`
`H2.9a=(f/.2)^8/(1+(f/.2)^8)`
`plot(f,h2.9a,type='l')`

we obtain the plot



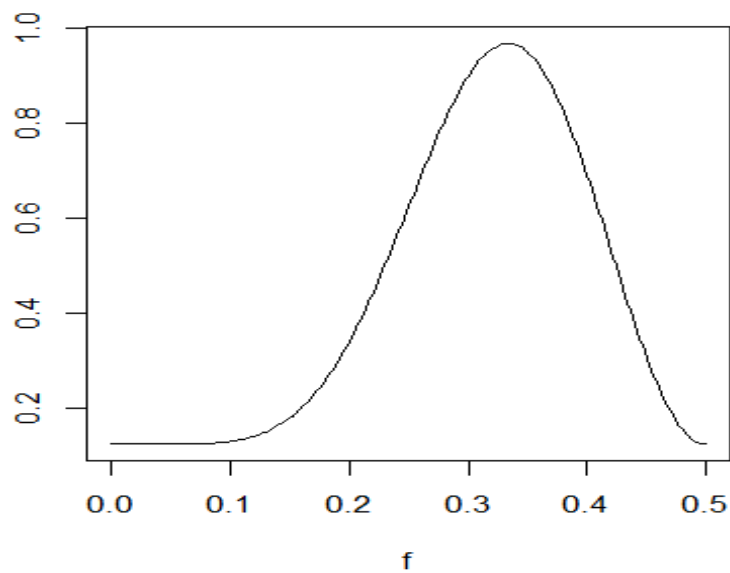
It can be seen that this is a high-pass filter with cutoff at about $f=0.2$.

(b) It can be seen that $|H(e^{-2\pi if})|^2 = (1 + (2\sin(2\pi f) - \sin(4\pi f))^2) / 8$

Using the code

```
ff=1:251
f=(ff-1)/500
H2.9b=(1+(2*sin(2*pi*f)-sin(4*pi*f))^2)/8
plot(f,H2.9b,type='l')
```

we obtain the plot below which shows that the filter is band-pass, passing mostly frequencies from $f=0.25$ to $f=0.45$



CHAPTER 3 – Problem Solutions

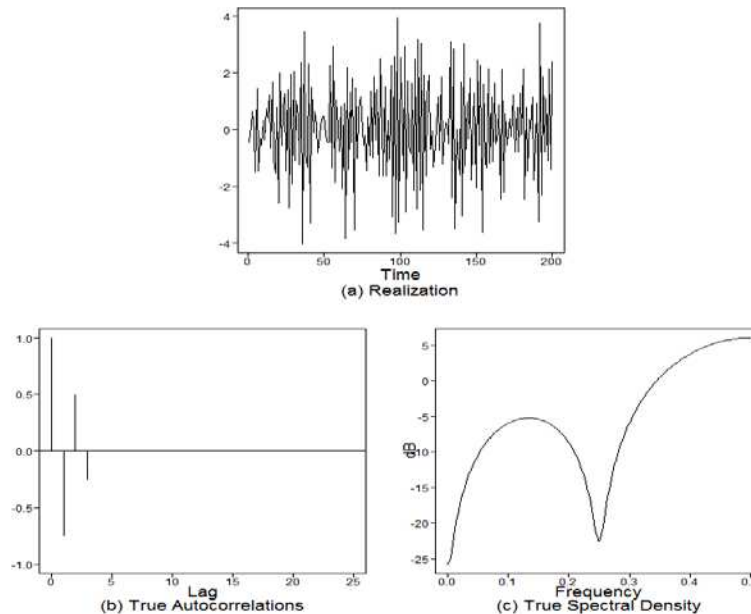
Applied Problems

Problem 3.1

(a,b,c) Using the code

```
x=plotts.true.wge(n=200,theta=c(.95,-.9,.855))
```

We obtained the plots

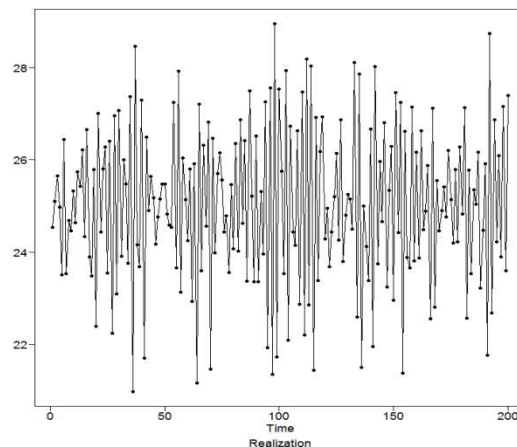


To obtain and plot a realization with mean 25 I used the commands

```
x25=mean(x$data)
```

```
plotts.wge(x25)
```

and obtained the following is the data with $\mu = 25$. The other two plots do not change with the mean change.



The realization shows high frequency behavior with oscillation back and forth across the mean of 25. The autocorrelations show only three non-zero autocorrelations. The fact that ρ_1 is fairly large negative and ρ_2 is positive, causes the up-and-down oscillatory behavior. The spectral density shows power at $f=0.5$ which