

***Finite Element Method Basic Concepts and
Applications with MATLAB MAPLE and COMSOL
3rd edition Pepper Solutions Manual***

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2.1 Fill in the details leading to Eq. (2.17) and use it to find the weighted residuals formulation of Example 2.1

Solution:

a)

$$\int_0^L \phi(x) \left[-K \frac{d^2 T}{dx^2} - Q \right] dx = 0 \quad (Eq. 2.13)$$

Integration by parts: $\int_0^L u dv = u v \Big|_0^L - \int_0^L v du$. Set $u = \phi(x)$ and $dv = -K \frac{d^2 T}{dx^2}$, then

$v = -K \frac{dT}{dx}$ and $du = \frac{d\phi}{dx} dx$ and we have

$$\begin{aligned} \int_0^L \phi(x) \left[-K \frac{d^2 T}{dx^2} - Q \right] dx &= \phi(x) \left(-K \frac{dT}{dx} \right) \Big|_0^L - \int_0^L \left(-K \frac{dT}{dx} \right) \frac{d\phi}{dx} dx \\ &= \int_0^L K \frac{d\phi}{dx} \frac{dT}{dx} dx - K \phi \frac{dT}{dx} \Big|_0^L \end{aligned}$$

Substitute into Eq. 2.13 and we get

$$\int_0^L K \frac{d\phi}{dx} \frac{dT}{dx} dx - \int_0^L \phi Q dx - K \phi \frac{dT}{dx} \Big|_0^L = 0 \quad (Eq. 2.17)$$

b)

$$T(0) = T_0 \quad \text{and} \quad -K \frac{dT}{dx} \Big|_{x=L} = h(T - T_\infty)$$

Replacing in Eq. 2.17

$$\int_0^L K \frac{d\phi}{dx} \frac{dT}{dx} dx - \int_0^L \phi Q dx - \phi(L) h(T(L) - T_\infty) = 0$$

2.2 Use Eq. (2.17) to find the weighted residuals formulation of Eqs. (2.1) – (2.3) in terms of the heat flux q .

Solution:

$$\int_0^L K \frac{d\phi}{dx} \frac{dT}{dx} dx - \int_0^L \phi Q dx + \phi \left(-K \frac{dT}{dx} \right) \Big|_0^L = 0$$

The heat flux term at $x = L$ must be zero, and $-K \frac{dT}{dx} \Big|_{x=0} = q$ Hence

$$\int_0^L K \frac{d\phi}{dx} \frac{dT}{dx} dx = \int_0^L \phi Q dx + \phi(0) q$$

2.3 Obtain expression (2.30).

Solution:

The weighted residuals form for the second term is

$$\int_0^L \phi(x) \left[-K \frac{d^2 T}{dx^2} - Q \right] dx = 0$$

After integration by parts on the first term

$$\int_0^L K \frac{d\phi}{dx} \frac{dT}{dx} dx - \int_0^L \phi Q dx + \phi \left(-K \frac{dT}{dx} \right) \Big|_{L/2}^L = 0$$

Note: The boundary terms are set to zero if no fluxes are prescribed.

The functions $\phi_i(x)$ for element 2 are

$$\phi_2(x) = \frac{2}{L}(L-x)$$

$$\phi_3(x) = \frac{2}{L}(x-L/2)$$

Let
$$T(x) = \frac{2}{L}(L-x)a_2 + \frac{2}{L}(x-L/2)a_3 \quad (\text{Eq. 2.19})$$

Using Eq. 2.28

$$\int_{L/2}^L \begin{bmatrix} -\frac{2}{L} \\ \frac{2}{L} \end{bmatrix} \begin{bmatrix} -\frac{2}{L} & \frac{2}{L} \end{bmatrix} dx \begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} - Q \int_{L/2}^L \begin{Bmatrix} \frac{2}{L}(L-x) \\ \frac{2}{L}(x-L/2) \end{Bmatrix} dx - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Or

$$\frac{4K}{L^2} \int_{L/2}^L \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx \begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} - \frac{2Q}{L} \int_{L/2}^L \begin{Bmatrix} (L-x) \\ (x-L/2) \end{Bmatrix} dx - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Integrating we have

$$\frac{2K}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} - \frac{QL}{4} \int_{L/2}^L \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} dx - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

2.4 Consider the equation

$$\frac{d^2 u}{dx^2} + u + x = 0, \quad 0 < x < 1$$

with $u(0) = u(1) = 0$. Assume an approximation for $u(x) = a_1 \phi_1(x)$ with $\phi_1(x) = x(1 - x)$.

The residual is

$$R(u, x) = -2a_1 + a_1[x(1 - x)] + x$$

and the weight $W_1(x) = \phi_1(x) = x(1 - x)$. Find the solution from the following integral

$$\int_0^1 W(x) R(u, x) dx = 0$$

.Solution:

$$\int_0^1 W(x) R(u, x) dx = \int_0^1 x(1 - x)(-2a_1 + a_1x(1 - x) + x) dx = 0$$

Or

$$\int_0^1 W(x) R(u, x) dx = \int_0^1 (-2a_1x + (3a_1 + 1)x^2 - 2a_1 + 1)x^3 + a_1x^4) dx = 0$$

$$-a_1 + \frac{1}{3}(3a_1 + 1) - \frac{1}{4}(2a_1 + 1) + \frac{1}{5}a_1 = -\frac{3}{10}a_1 + \frac{1}{12} = 0$$

$$a_1 = \frac{5}{18} \quad \text{therefore} \quad u(x) \approx \frac{5}{18}(x - x^2)$$

Compare with the exact solution given by

$$u^*(x) = \frac{\sin x}{\sin 1} - x$$

Evaluating at $x=1/2$ $u(1/2) = 0.069444$ less than 5% error.

$$u^*(1/2) = 0.069747$$

2.5 Use the weak statement formulation to find solutions to the equation

$$\frac{d^2 u}{dx^2} = 1, \quad 0 < x < 1$$

$$u(0) = 0, \quad \left. \frac{du}{dx} \right|_{x=1} = 1$$

(a) Using linear interpolation functions, as explained in Section 2.3, use (i) one element and (ii) two elements.

(b) Using simple polynomial functions that satisfy the boundary conditions at the left-hand side, i.e.,

(i) $u(x) = a_1 x$, hence $\phi_1(x) = W_1(x) = x$

(ii) $u(x) = a_1 x + a_2 x^2$, hence $\phi_1(x) = W_1(x) = x$,

$$\phi_2(x) = W_2(x) = x^2.$$

(c) Using circular functions that satisfy the boundary conditions at the left-hand side, i.e.,

(i) $u(x) = a_1 \sin(\pi/2 Lx)$

(ii) $u(x) = a_1 \sin(\pi/2 Lx) + a_2 \sin(3\pi/2 Lx)$

(d) Find the analytical solution and compare with results from (a), (b) and (c).

Solution

a) The residual function is $R(u, x) = \frac{d^2 u}{dx^2} + 1$ The weak form after integration by parts is

$$\int_0^1 \left(\frac{dW}{dx} \frac{du}{dx} + W \right) dx - W \frac{du}{dx} \Big|_0^1 = 0$$

i)

One element solution $\phi_1 = 1 - x \quad \phi_2 = x$

$$u(x) = (1 - x) a_1 + x a_2$$

$$\int_0^1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} dx \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} + \int_0^1 \begin{bmatrix} 1 - x \\ x \end{bmatrix} dx \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Integrating

$$a_1 - a_2 = -1/2$$

$$-a_1 + a_2 = 1/2$$

$$\text{since } a_1 = 0 \quad \text{then } a_2 = 1/2 \quad \text{or } u(x) = x/2$$

ii)

Two element solution. Element 1:

$$\phi_1 = 1 - 2x \quad \phi_2 = 2x$$

$$\int_0^{1/2} \begin{bmatrix} -2 \\ 2 \end{bmatrix} \begin{bmatrix} -2 & 2 \end{bmatrix} dx \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} + \int_0^{1/2} \begin{bmatrix} 1-2x \\ 2x \end{bmatrix} dx \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{or} \quad \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} -1/4 \\ -1/4 \end{Bmatrix}$$

Element 2

$$\phi_2 = 2(1-x) \quad \phi_3 = 2(x-1/2)$$

$$\int_{1/2}^1 \begin{bmatrix} -2 \\ 2 \end{bmatrix} \begin{bmatrix} -2 & 2 \end{bmatrix} dx \begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} + \int_{1/2}^1 \begin{bmatrix} 2(1-x) \\ 2(x-1/2) \end{bmatrix} dx \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{or} \quad \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} -1/4 \\ 3/4 \end{Bmatrix}$$

Assemble

the

elements

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} -1/4 \\ -1/2 \\ 3/4 \end{Bmatrix} \quad \text{since } a_1 = 0 \text{ we get } a_2 = 1/8 \text{ and } a_3 = 1/2$$

$$u(x) = \begin{cases} x/4 & 0 \leq x \leq 1/2 \\ \frac{3x-1}{4} & 1/2 \leq x \leq 1 \end{cases}$$

b)

Using polynomials

i)

$$\phi_1 = x \quad u(x) = a_1 x$$

$$\int_0^1 (1 \cdot a_1 + x) dx - 1 = 0, \quad a_1 = 1/2, \quad u(x) = x/2, \text{ same as using one element.}$$

ii)

$$\phi_1 = x, \quad \phi_2 = x^2, \quad u(x) = a_1 x + a_2 x^2$$

$$\text{For } W = \phi_1 \quad \int_0^1 [1 \cdot (a_1 + 2a_2 x) + x] dx - 1 = 0$$

$$\text{For } W = \phi_2 \quad \int_0^1 [2x \cdot (a_1 + 2a_2 x) + x^2] dx - 1 = 0$$

The system becomes

$$a_1 + a_2 = 1/2$$

$$a_1 + \frac{4}{3}a_2 = 2/3 \quad \text{or} \quad a_1 = 0, \quad a_2 = 1/2 \quad \text{and} \quad u(x) = x^2/2$$

c)

Using circular functions

i)

$$\phi_1 = \sin\left(\frac{\pi}{2}x\right) \quad , \quad u(x) = a_1 \sin\left(\frac{\pi}{2}x\right) \quad , \quad \phi'_1 = \frac{\pi}{2} \cos\left(\frac{\pi}{2}x\right)$$

$$\int_0^1 \left[\frac{\pi^2}{4} a_1 \cos^2\left(\frac{\pi}{2}x\right) + \sin\left(\frac{\pi}{2}x\right) \right] dx - \sin\left(\frac{\pi}{2}x\right) \frac{du}{dx} \Big|_0^1 = 0$$

Use the identities $\int \cos^2 \alpha x dx = \frac{1}{2\alpha} (\sin \alpha x \cos \alpha x - \alpha x)$ and $\int \sin\left(\frac{\pi}{2}x\right) dx = -\frac{2}{\pi} \cos\left(\frac{\pi}{2}x\right)$. Then

$$\frac{\pi^2 a_1}{4} \left(\sin\left(\frac{\pi}{2}x\right) \cos\left(\frac{\pi}{2}x\right) - \frac{x}{2} \right) \Big|_0^1 + \left(-\frac{2}{\pi} \cos\left(\frac{\pi}{2}x\right) \right) \Big|_0^1 - \sin\left(\frac{\pi}{2}x\right) \frac{du}{dx} \Big|_0^1 = 0 \quad \text{or}$$

$$\frac{\pi^2 a_1}{8} + \frac{2}{\pi} - 1 = 0 \quad \text{hence} \quad a_1 \approx 0.295 \quad \text{and} \quad u(x) = 0.295 \sin\left(\frac{\pi}{2}x\right)$$

ii)

$$\phi_2 = \sin\left(\frac{3\pi}{2}x\right) \quad , \quad u(x) = a_1 \sin\left(\frac{\pi}{2}x\right) + a_2 \sin\left(\frac{3\pi}{2}x\right).$$

$$\text{For } W = \phi_1 : \quad \int_0^1 \left[\frac{\pi^2}{4} \cos\left(\frac{\pi}{2}x\right) \left(a_1 \cos\left(\frac{\pi}{2}x\right) + a_2 \cos\left(\frac{3\pi}{2}x\right) \right) + \sin\left(\frac{\pi}{2}x\right) \right] dx - \sin\left(\frac{\pi}{2}x\right) \frac{du}{dx} \Big|_0^1 = 0$$

$$\text{For } W = \phi_2 : \quad \int_0^1 \left[3 \frac{\pi^2}{4} \cos\left(\frac{3\pi}{2}x\right) \left(a_1 \cos\left(\frac{\pi}{2}x\right) + a_2 \cos\left(\frac{3\pi}{2}x\right) \right) + \sin\left(\frac{3\pi}{2}x\right) \right] dx - \sin\left(\frac{3\pi}{2}x\right) \frac{du}{dx} \Big|_0^1 =$$

$$\frac{\pi^2}{8} a_1 + 0 \cdot a_2 + \frac{2}{\pi} - 1 = 0 \quad (a_1 \text{ is unchanged})$$

$$0 \cdot a_1 + \frac{9\pi^2}{8} a_2 + \frac{2}{3\pi} + 1 = 0 \quad , \quad a_2 \approx -0.109 \quad \text{and}$$

$$u(x) \approx 0.295 \sin\left(\frac{\pi}{2}x\right) - 0.109 \sin\left(\frac{3\pi}{2}x\right)$$

d)

Analytical

solution

$$\frac{d^2 u}{dx^2} = 1 \quad , \text{integrating} \quad u(x) = \frac{x^2}{2} + c_1 x + c_2 \quad \text{and applying boundary conditions} \quad u^*(x) = \frac{x^2}{2}.$$

Notice that all the approximations are exact at $x = 1/2$, and the Galerkin approximations using x and x^2 both yield the exact solution. Also notice that the boundary condition at $x = 1$ is satisfied only approximately. In case a), the first solution gives $\left. \frac{du}{dx} \right|_{x=1} = \frac{1}{2}$ and the second solution gives

$\left. \frac{du}{dx} \right|_{x=1} = \frac{3}{4}$ which is getting better. This point is discussed in more detail in problem 2.6.

2.6 Subdivide the interval $0 \leq x \leq L$ into 10 linear elements, construct the element equations for four consecutive elements starting with element four, and show that none of the parameters a_i is related to more than two additional parameters in the equations. The only ones involved being a_{i-1} and a_{i+1} (as a consequence of this, the final system of linear equations will involve a tri-diagonal matrix which is easy to solve).

Solution:

For an element $e_k = \{x/x_k \leq x \leq x_{k+1}\}$ the element stiffness matrix is obtained from

$\int_{x_k}^{x_{k+1}} \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx$. All the stiffness matrices must be equal, so let's calculate the first one.

$$\phi_1 = 1 - \frac{10}{L}x \quad \phi_2 = \frac{10}{L}x \quad \text{thus} \quad \int_0^{L/10} K \begin{bmatrix} -\frac{10}{L} \\ \frac{10}{L} \end{bmatrix} \begin{bmatrix} -\frac{10}{L} & \frac{10}{L} \end{bmatrix} dx = \frac{10K}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\text{and after assembling} \quad \frac{10K}{L} \begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_9 \\ a_{10} \\ a_{11} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Note that each equation i involves only three unknowns a_{i-1} , a_i and a_{i+1} , so the stiffness matrix is tri-diagonal.

2.7 The Rayleigh-Ritz formulation for the problem of solving Eq. (2.1) with boundary conditions (2.2) and (2.3) can be stated as: Minimize the functional

$$F(T) = \int_0^L \left\{ \frac{K}{2} \left[\frac{dT}{dx} \right]^2 - QT \right\} dx - Tq \Big|_{x=0}$$

over all functions $T(x)$ with square integrable first derivatives that satisfy Eq. (2.3) at $x = L$.

Approximate $T(x)$ using two linear elements as we did before and replace into Eq. (2.1) to obtain $F(T) \cong F(a_1, a_2, a_3)$. Now minimize $F(a_1, a_2, a_3)$ as a function of three variables. Show that the final system of equations is identical to Eq. (2.31), thus, in this case, the Galerkin method and the Rayleigh-Ritz method are equivalent.

Solution:

Element1:

$$\phi_1 = 1 - \frac{2}{L}x \quad \phi_2 = \frac{2}{L}x$$

$$\int_0^{L/2} \frac{K}{2} \left(-\frac{2}{L}a_1 + \frac{2}{L}a_2 \right)^2 - Q \left[\left(1 - \frac{2}{L}x \right) a_1 + \frac{2}{L}x a_2 \right] dx - a_1 q = \frac{K}{L} (a_2 - a_1)^2 - \frac{QL}{4} (a_1 + a_2) - a_1 q \quad (1)$$

Element2:

$$\phi_2 = 2 - \frac{2}{L}x \quad \phi_3 = \frac{2}{L}x - 1$$

$$\int_{L/2}^L \frac{K}{2} \left(-\frac{2}{L}a_2 + \frac{2}{L}a_3 \right)^2 - Q \left[\left(2 - \frac{2}{L}x \right) a_2 + \left(\frac{2}{L}x - 1 \right) a_3 \right] dx - a_1 q = \frac{K}{L} (a_3 - a_2)^2 - \frac{QL}{4} (a_2 + a_3) \quad (2)$$

$$F(T) = (1) + (2) = \frac{K}{L} (a_2 - a_1)^2 + \frac{K}{L} (a_3 - a_2)^2 - \frac{QL}{4} (a_1 + 2a_2 + a_3) - a_1 q$$

$$F(T) = F(a_1, a_2, a_3) = \frac{K}{L} (a_1^2 - 2a_1a_2 + 2a_2^2 - 2a_2a_3 + a_3^2)$$

The conditions for a minimum are $\frac{\partial F}{\partial a_1} = \frac{\partial F}{\partial a_2} = \frac{\partial F}{\partial a_3} = 0$ thus

$$\frac{\partial F}{\partial a_1} = \frac{2K}{L} (a_1 - a_2) - \frac{QL}{4} - q = 0$$

$$\frac{\partial F}{\partial a_2} = \frac{2K}{L} (-a_1 + 2a_2 - a_3) - \frac{QL}{2} = 0$$

$$\frac{\partial F}{\partial a_3} = \frac{2K}{L} (a_3 - a_2) - \frac{QL}{4} = 0$$

In Matrix form $\frac{2K}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{QL}{4} \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix} + \begin{Bmatrix} q \\ 0 \\ 0 \end{Bmatrix}$ which is identical to Eq. 2.31.

2.8 Let us consider once more Problem 2.3 above. Assume that $u(x) = a_1 x^2$ and $W_1(x) = x^2$, but this time use Eq. (2.11) and proceed as in Problem 2.2 in order to obtain a solution. Show that this solution is identical to that obtained in Problem 2.3(b)(ii), and hence, the process of integration by parts does not introduce any changes.

Solution:

$$\int_0^1 \left[W_1 \left(\frac{d^2 u}{dx^2} - 1 \right) \right] dx = \int_0^1 x^2 (2a_1 - 1) dx = 0, \quad \left(\frac{2}{3} x^3 a_1 - \frac{x^3}{3} \right) \bigg|_0^1 = 0, \quad \frac{2}{3} a_1 - \frac{1}{3} = 0, \quad a_1 = 1/2, \quad u(x) = \frac{x^2}{2}$$

2.9 In example 2.2, first find the exact analytical solution, then calculate two more Galerkin approximations using

Solution:

The analytical solution is $u(x) = \frac{1}{2}(x - x^2)$. The Galerkin approximation is

$$\int_0^1 \left(\frac{dW}{dx} \frac{du}{dx} - W \right) dx = 0.$$

$$u(x) = a_1 \sin \pi x + a_2 \sin 2\pi x, \quad W_1 = \sin \pi x, \quad W_2 = \sin 2\pi x$$

$$a) \quad W_1: \int_0^1 \left[\pi^2 \cos \pi x (a_1 \cos \pi x + 2a_2 \cos 2\pi x) - \sin \pi x \right] dx = 0$$

$$W_2: \int_0^1 \left[2\pi^2 \cos 2\pi x (a_1 \cos \pi x + 2a_2 \cos 2\pi x) - \sin 2\pi x \right] dx = 0 \quad \text{or}$$

$$\frac{\pi^2}{2} a_1 + 0 \cdot a_2 - \frac{2}{\pi} = 0, \quad a_1 = \frac{4}{\pi^3}, \quad \text{and} \quad 0 \cdot a_1 + 2\pi^2 a_2 + 0 = 0, \quad a_2 = 0$$

$$u(x) = \frac{4}{\pi^3} \sin \pi x$$

$$b) \quad u(x) = a_1 \sin \pi x + a_2 \sin 2\pi x + a_3 \sin 3\pi x, \quad W_1 = \sin \pi x, \quad W_2 = \sin 2\pi x, \quad W_3 = \sin 3\pi x$$

We only need to calculate a_3 , a_1 and a_2 remain the same due to the orthogonality of the sine functions.

$$W_3: \int_0^1 \left[3\pi^2 \cos 3\pi x (a_1 \cos \pi x + 2a_2 \cos 2\pi x + 3a_3 \cos 3\pi x) - \sin 3\pi x \right] dx = 0$$

$$\frac{9\pi^2}{2} a_3 - \frac{2}{3\pi} = 0, \quad a_3 = \frac{4}{27\pi^3}, \quad u_3(x) = \frac{4}{3\pi^3} \left(\sin \pi x + \frac{1}{27} \sin 3\pi x \right)$$

2.10 Derive Eq. (2.32) from Eq. (2.31).

Eqs. 2.32 $a_1 - a_2 = \frac{qL}{2K} + \frac{QL^2}{8K}$, $-a_1 + 2a_2 = \frac{QL^2}{4K} + T_L$ adding $a_2 = \frac{qL}{2K} + \frac{3QL^2}{8K} + T_L$ and

substituting into the first equation $a_1 = \frac{qL}{K} + \frac{QL^2}{2K} + T_L$

2.11 Fill in the details in Example 2.3 and verify the solution.

Solution:

The weak form is $\int_0^1 \left(\frac{d\phi}{dx} \frac{dT}{dx} + W \right) dx - \int_0^1 \phi dx = 0$ Use the Galerkin approximation

$u(x) = a_1 \sin \pi x$, $\phi(x) = \sin \pi x$, $du = a_1 \pi \cos \pi x dx$, $d\phi = \pi \cos \pi x dx$. Replace in the integrals to get

$\int_0^1 (\pi \cos \pi x) (a_1 \pi \cos \pi x) dx = \int_0^1 \sin \pi x dx$ and using the trigonometric identity

$$\int_0^1 \cos^2 \pi x dx = \int_0^1 \frac{1}{2} (1 + \cos 2\pi x) dx \text{ we get } a_1 = \frac{4}{\pi^3}.$$

2.12 Show that if the expressions for ϕ_i given by Eq. (2.20) are used in Eq. (2.19), then $a_i = T_i$.

Solution:

Eq. 2.19 is $T(x) = \phi_i(x)a_i + \phi_{i+1}(x)a_{i+1} \quad x_i \leq x \leq x_{i+1}$ where

$$\phi_i(x) = \frac{x_{i+1} - x}{x_{i+1} - x_i}, \text{ and } \phi_{i+1}(x) = \frac{x - x_i}{x_{i+1} - x_i}$$

Substituting $T(x) = \left[\frac{x_{i+1} - x}{x_{i+1} - x_i} \right] a_i + \left[\frac{x - x_i}{x_{i+1} - x_i} \right] a_{i+1}$ and evaluating at the nodes

$$T(x_i) = a_i = T_i \text{ and } T(x_{i+1}) = a_{i+1} = T_{i+1}.$$

2.13 Find the weak formulation of Eq. (2.1) with the boundary condition

$$-K \frac{dT}{dx} \Big|_{x=0} = h(T - T_{\infty}) \text{ and } -K \frac{dT}{dx} \Big|_{x=L} = q.$$

Solution:

The general form is given in Eq. 2.17 $\int_0^L \frac{d\phi}{dx} \frac{dT}{dx} dx - \int_0^L \phi Q dx - K \phi \frac{dT}{dx} \Big|_0^L = 0$

Substituting the flux boundary conditions we get

$$\int_0^L \frac{d\phi}{dx} \frac{dT}{dx} dx - \int_0^L \phi Q dx + \phi(L)q - \phi(0)h(T(0) - T_{\infty}) = 0$$

2.14 Using two linear elements solve the equations in problem (2.13) with $L = 1 \text{ m}$, $K = 200 \text{ W/mK}$, $Q = 100 \text{ W/m}^3$, $h = 150 \text{ W/m}^2 \text{ K}$, $T_\infty = 100^\circ\text{C}$ and $q = 2,000 \text{ W/m}^2$.

Solution:

Using Eqs. 2.19 and 2.30 $\frac{2K}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} - \frac{QL}{4} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} - h \begin{Bmatrix} T_1 - T_\infty \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ Substitute the

parameters values 1) $\begin{bmatrix} 250 & -400 \\ -400 & 400 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} -14750 \\ 25 \end{Bmatrix}$ 2) $\begin{bmatrix} 400 & -400 \\ -400 & 400 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 25 \\ 25 - q \end{Bmatrix}$

assembling

$$1) \begin{bmatrix} 250 & -400 & 0 \\ -400 & 800 & -400 \\ 0 & -400 & 400 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} -14750 \\ 50 \\ 25 - q \end{Bmatrix} \quad \text{With solution} \quad \begin{aligned} T_1 &= 97.833 + 0.0066667q \\ T_2 &= 98.021 + 0.0041667q \\ T_3 &= 98.084 + 0.0016667q \end{aligned}$$