

Advanced Engineering Mathematics with MATLAB
4th edition Duffy Solutions Manual

SKU: 9781498739641-SOLUTIONS

```

% take the first time step

x(1) = 0; t(1) = 0; t(2) = deltat;
x(2) = x(1) + deltat - b * deltat * sign(x(1)) * abs(x(1))^beta;
sum = x(1) + x(2);

% take the remaining time steps

for k = 2:K
    t(k) = t(k-1) + deltat;
    x(k) = x(k-1) + deltat ...
        - b * deltat * sign(x(k-1)) * abs(x(k-1))^beta ...
        - deltat * deltat * sum;
    sum = sum + x(k);
end

% plot the results

subplot(2,2,n), plot(t,x); xlabel('time','FontSize',20);
ylabel('x(t)','FontSize',20); legend(['B = ',num2str(b)])

end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

```

Section 2.0

1. Since the second solution is $y_2(x) = u(x)$, $y_2'(x) = u'(x)$ and $y_2''(x) = u''(x)$. Consequently, substituting these values of $y_2(x)$, $y_2'(x)$, and $y_2''(x)$ into the differential equation, we find that $xu'' + 2u' = 0$. Therefore,

$$\frac{u''(x)}{u'(x)} = -\frac{2}{x}, \quad u'(x) = Cx^{-2}.$$

Thus, $u(x) = A/x$, and the second solution is $y_2(x) = A/x$.

2. Since the second solution is $y_2(x) = u(x)e^x$,

$$y_2'(x) = u'(x)e^x + u(x)e^x, \quad \text{and} \quad y_2''(x) = u''(x)e^x + 2u'(x)e^x + u(x)e^x.$$

Consequently, substituting these values of $y_2(x)$, $y_2'(x)$, and $y_2''(x)$ into the differential equation, we find that $u'' + 3u' = 0$. Therefore,

$$\frac{u''(x)}{u'(x)} = -3, \quad u'(x) = Ce^{-3x}.$$

Thus, $u(x) = Ae^{-3x}$, and the second solution is $y_2(x) = Ae^{-2x}$.

3. Since the second solution is $y_2(x) = xu(x)$,

$$y_2'(x) = xu'(x) + u(x), \quad \text{and} \quad y_2''(x) = xu''(x) + 2u'(x).$$

Consequently, substituting these values of $y_2(x)$, $y_2'(x)$, and $y_2''(x)$ into the differential equation, we find that $xu'' + 6u' = 0$. Therefore,

$$\frac{u''(x)}{u'(x)} = -\frac{6}{x}, \quad u'(x) = Cx^{-6}.$$

Thus, $u(x) = Ax^{-5}$, and the second solution is $y_2(x) = Ax^{-4}$.

4. Since the second solution is $y_2(x) = e^x u(x)$,

$$y_2'(x) = e^x u(x) + e^x u'(x), \quad \text{and} \quad y_2''(x) = e^x u(x) + 2e^x u'(x) + e^x u''(x).$$

Consequently, substituting these values of $y_2(x)$, $y_2'(x)$, and $y_2''(x)$ into the differential equation, we find that $xu'' + (x-1)u' = 0$. Therefore,

$$\frac{u''(x)}{u'(x)} = \frac{1}{x} - 1, \quad u'(x) = -Axe^{-x}.$$

Thus, $u(x) = A(x+1)e^{-x}$, and the second solution is $y_2(x) = A(x+1)$.

5. Since the second solution is $y_2(x) = (x-1)u(x)$,

$$y_2'(x) = (x-1)u'(x) + u(x), \quad \text{and} \quad y_2''(x) = (x-1)u''(x) + 2u'(x).$$

Consequently, substituting these values of $y_2(x)$, $y_2'(x)$, and $y_2''(x)$ into the differential equation, we find that $x(2-x)(x-1)u'' + 2u' = 0$. Therefore,

$$\frac{u''(x)}{u'(x)} = \frac{2}{x(x-2)(x-1)} = \frac{1}{x} + \frac{1}{x-2} - \frac{2}{x-1},$$

and

$$u'(x) = A \frac{x(x-2)}{(x-1)^2} = A \left[1 - \frac{1}{(x-1)^2} \right].$$

Thus, $u(x) = Ax + A/(x-1)$, and the second solution is $y_2(x) = A(x^2 - x + 1)$.

6. Since the second solution is $y_2(x) = u(x) \sin^3(x)$,

$$y_2'(x) = 3 \sin^2(x) \cos(x) u(x) + \sin^3(x) u'(x),$$

and

$$y_2''(x) = 6 \sin(x) \cos^2(x) u(x) - 3 \sin^3(x) u(x) \\ + 6 \sin^2(x) \cos(x) u'(x) + \sin^3(x) u''(x).$$

Consequently, substituting these values of $y_2(x)$, $y_2'(x)$, and $y_2''(x)$ into the differential equation, we find that

$$\sin(x) \cos(x) u'' + [6 \cos^2(x) + \sin^2(x)] u' = 0.$$

Therefore,

$$\frac{u''(x)}{u'(x)} = -6 \cot(x) - \tan(x), \quad u'(x) = -\frac{A \cos(x)}{5 \sin^6(x)}.$$

Thus, $u(x) = A/\sin^5(x)$, and the second solution is $y_2(x) = A/\sin^2(x)$.

7. Since the second solution is $y_2(x) = u(x) \cos(x)/\sqrt{x}$,

$$y_2'(x) = -\frac{\sin(x)}{\sqrt{x}} u(x) - \frac{\cos(x)}{2x\sqrt{x}} u(x) + \frac{\cos(x)}{\sqrt{x}} u'(x)$$

and

$$y_2''(x) = -\frac{\cos(x)}{\sqrt{x}} u(x) + \frac{3 \cos(x)}{4x^2\sqrt{x}} u(x) + \frac{\sin(x)}{x\sqrt{x}} u(x) \\ - \frac{\cos(x)}{x\sqrt{x}} u'(x) - \frac{2 \sin(x)}{\sqrt{x}} u'(x) + \frac{\cos(x)}{\sqrt{x}} u''(x).$$

Consequently, substituting these values of $y_2(x)$, $y_2'(x)$, and $y_2''(x)$ into the differential equation, we find that $\cos(x)u'' - 2 \sin(x)u' = 0$. Therefore,

$$\frac{u''(x)}{u'(x)} = \frac{2 \sin(x)}{\cos(x)}, \quad u'(x) = A \sec^2(x).$$

Thus, $u(x) = A \tan(x)$, and the second solution is $y_2(x) = A \sin(x)/\sqrt{x}$.

8. Since the second solution is $y_2(x) = u(x)e^{-bx^2/2}$

$$y_2'(x) = -bx e^{-bx^2/2} u(x) + e^{-bx^2/2} u'(x),$$

and

$$y_2''(x) = (b^2 x^2 - b) e^{-bx^2/2} u(x) - 2bx e^{-bx^2/2} u'(x) + u''(x) e^{-bx^2/2}.$$

Consequently, substituting these values of $y_2(x)$, $y_2'(x)$, and $y_2''(x)$ into the differential equation, we find that $u'' + (a - 2bx)u' = 0$. Therefore,

$$\frac{u''(x)}{u'(x)} = 2bx - a.$$

Thus, $u(x) = \int^x e^{b\xi^2 - a\xi} d\xi$, and the second solution is

$$y_2(x) = e^{-bx^2/2} \int^x e^{b\xi^2 - a\xi} d\xi.$$

9. Letting $v = y'$, we can rewrite the differential equation $y(dv/dy) = v^2$. Assuming $v \neq 0$, we can integrate this equation to give $v = C_1 y$. Therefore, $dy/dx = C_1 y$. Integrating this, we obtain the final answer $y(x) = C_2 e^{C_1 x}$.

10. Letting $v = y'$, we can rewrite the differential equation $dv/dy = 2y$. Assuming $v \neq 0$, we can integrate this equation to give $v = y^2 + C_1$. Because $v(1) = 1$, $C_1 = 0$ and $dy/dx = y^2$. Integrating this equation, $1/y = C_2 - x$. Again, using the initial conditions, $y(x) = 1/(1 - x)$.

11. Letting $v = y'$, we can rewrite the differential equation $yv(dv/dy) = v + v^2$. Assuming $v \neq 0$, we can integrate this equation to give $1 + v = C_1 y$. Substituting for v , $dy/dx = C_1 y - 1$. Integrating this equation, $y = (1 + C_2 e^{C_1 x}) / C_1$.

12. Letting $v = y'$, we can rewrite the differential equation $2yv(dv/dy) = 1 + v^2$. Assuming $v \neq 0$, we can integrate this equation to give $1 + v^2 = C_1 y$. Substituting for v , $dy/dx = \sqrt{C_1 y - 1}$. Integrating this equation, $y = [1 + (C_1 x + C_2)^2 / 4] / C_1$.

13. Letting $v = y'$, we can rewrite the differential equation $v(dv/dy) = e^{2y}$ with $v(0) = 1$. We can integrate this equation and find $v = e^y$. Substituting for v , $dy/dx = e^y$. Integrating this equation, $e^{-y} = C - x$, or $y = -\ln|1 - x|$.

14. We begin by integrating once and using the initial conditions, $y'' = \frac{3}{2}y^2$. Letting $v = y'$, we can rewrite this differential equation $v(dv/dy) = \frac{3}{2}y^2$. Integrating this equation, we have $v = y^{3/2}$. Substituting for v , $dy/dx = y^{3/2}$. Integrating this equation, $y = 4/(2 - x)^2 = 4/(x - 2)^2$.

15. If we define $z = 1/v$, the Bernoulli equation becomes $z' + z/x = -\frac{1}{2}$. Its solution is $z = -A^2/x - x/4$. Therefore, $y' = v = -4x/(x^2 + 4A^2)$. Integration yields the final answer $y(x) = B - 2\ln(x^2 + 4A^2)$.

16. First we compute

$$y'(x) = u'(x) \exp\left[-\frac{1}{2} \int^x \frac{a_1(\xi)}{a_2(\xi)} d\xi\right] - \frac{a_1}{2a_2} u(x) \exp\left[-\frac{1}{2} \int^x \frac{a_1(\xi)}{a_2(\xi)} d\xi\right],$$

and

$$\begin{aligned} y''(x) = & u''(x) \exp\left[-\frac{1}{2} \int^x \frac{a_1(\xi)}{a_2(\xi)} d\xi\right] - \frac{1}{2} \frac{d}{dx} \left(\frac{a_1}{a_2}\right) u(x) \exp\left[-\frac{1}{2} \int^x \frac{a_1(\xi)}{a_2(\xi)} d\xi\right] \\ & - \frac{a_1}{a_2} u'(x) \exp\left[-\frac{1}{2} \int^x \frac{a_1(\xi)}{a_2(\xi)} d\xi\right] + \frac{a_1^2}{4a_2^2} u(x) \exp\left[-\frac{1}{2} \int^x \frac{a_1(\xi)}{a_2(\xi)} d\xi\right]. \end{aligned}$$

Substituting $y(x)$, $y'(x)$, and $y''(x)$ into the original ordinary differential equation yields the final answer.

Section 2.1

1. The characteristic equation is $m^2 + 6m + 5 = (m+1)(m+5) = 0$. Therefore, the general solution is $y(x) = C_1 e^{-x} + C_2 e^{-5x}$.
2. The characteristic equation is $m^2 - 6m + 10 = (m-3+i)(m-3-i) = 0$. Therefore, the general solution is $y(x) = C_1 e^{3x} \cos(x) + C_2 e^{3x} \sin(x)$.
3. The characteristic equation is $m^2 - 2m + 1 = (m-1)^2 = 0$. Therefore, the general solution is $y(x) = C_1 e^x + C_2 x e^x$.
4. The characteristic equation is $m^2 - 3m + 2 = (m-1)(m-2) = 0$. Therefore, the general solution is $y(x) = C_1 e^{2x} + C_2 e^x$.
5. The characteristic equation is $m^2 - 4m + 8 = (m-2)^2 + 4 = 0$. Therefore, the general solution is $y(x) = C_1 e^{2x} \cos(2x) + C_2 e^{2x} \sin(2x)$.
6. The characteristic equation is $m^2 + 6m + 9 = (m+3)^2 = 0$. Therefore, the general solution is $y(x) = C_1 e^{-3x} + C_2 x e^{-3x}$.
7. The characteristic equation is $m^2 + 6m - 40 = (m+10)(m-4) = 0$. Therefore, the general solution is $y(x) = C_1 e^{-10x} + C_2 e^{4x}$.
8. The characteristic equation is $m^2 + 4m + 5 = (m+2)^2 + 1 = 0$. Therefore, the general solution is $y(x) = C_1 e^{-2x} \cos(x) + C_2 e^{-2x} \sin(x)$.
9. The characteristic equation is $m^2 + 8m + 25 = (m+4)^2 + 9 = 0$. Therefore, the general solution is $y(x) = e^{-4x} [C_1 \cos(3x) + C_2 \sin(3x)]$.
10. The characteristic equation is $4m^2 - 12m + 9 = (2m-3)^2 = 0$. Therefore, the general solution is $y(x) = e^{3x/2} (C_1 + C_2 x)$.
11. The characteristic equation is $m^2 + 8m + 16 = (m+4)^2 = 0$. Therefore, the general solution is $y(x) = C_1 e^{-4x} + C_2 x e^{-4x}$.
12. The characteristic equation is $m^3 + 4m^2 = m^2(m+4) = 0$. Therefore, the general solution is $y(x) = C_1 + C_2 x + C_3 e^{-4x}$.
13. The characteristic equation is $m^4 + 4m^2 = m^2(m^2 + 4) = 0$. Therefore, the general solution is $y(x) = C_1 + C_2 x + C_3 \cos(2x) + C_4 \sin(2x)$.
14. The characteristic equation is $m^4 + 2m^3 + m^2 = m^2(m+1)^2 = 0$. Therefore, the general solution is $y(x) = C_1 + C_2 x + C_3 e^{-x} + C_4 x e^{-x}$.

15. The characteristic equation is $m^3 + 8 = (m-2)(m+1+\sqrt{3})(m+1-\sqrt{3}) = 0$. Therefore, the general solution is

$$y(x) = C_1 e^{2x} + C_2 e^{-x} \cos(\sqrt{3}x) + C_3 e^{-x} \sin(\sqrt{3}x).$$

16. The characteristic equation is $m^4 - 3m^3 + 3m^2 - m = m(m-1)^3 = 0$. Therefore, the general solution is $y(x) = C_1 + (C_2 + C_3x + C_4x^2)e^x$.

17. Taking the derivative of the integro-differential equation,

$$\frac{d^2y}{dt^2} = \frac{A}{2\tau^2} \int_{-\infty}^t e^{-(t-x)/\tau} y(x) dx - \frac{A}{2\tau} y(t),$$

or

$$\frac{d^2y}{dt^2} + \frac{1}{\tau} \frac{dy}{dt} + \frac{A}{2\tau} y(t) = 0.$$

Try the solution $y(t) = Ce^{-\lambda t/2}$. Then

$$\frac{\lambda^2}{4} - \frac{\lambda}{2\tau} + \frac{A}{2\tau} = \frac{1}{4} \left[\left(\lambda - \frac{1}{\tau} \right)^2 + \frac{2A}{\tau} - \frac{1}{\tau^2} \right] = 0.$$

The roots to this equation are $\lambda_{\pm} = (1 \pm \sqrt{1 - 2A\tau})/\tau$. Therefore, the general solution is

$$y(t) = e^{-t/(2\tau)} \left\{ A \exp \left[t\sqrt{1 - 2A\tau}/(2\tau) \right] + B \exp \left[-t\sqrt{1 - 2A\tau}/(2\tau) \right] \right\}.$$

Section 2.2

1. The homogeneous solution to the equation is $x(t) = A \cos(5t) + B \sin(5t)$. Immediately, we have that $x(0) = A = 10$. Because $x'(0) = 5B = -10$, then $B = -2$. Therefore, $x(t) = 10 \cos(5t) - 2 \sin(5t)$. If we write the solution $x(t) = C \sin(5t + \varphi)$, then $C = \sqrt{100 + 4} = \sqrt{104} = 2\sqrt{26}$. On the other hand, $\varphi = \tan^{-1}(10/-2) = \tan^{-1}(-5) = 1.7682$. Our choice of angle is dictated by $A = C \sin(\varphi)$ and $B = C \cos(\varphi)$. Therefore the final answer is $x(t) = 2\sqrt{26} \sin(5t + 1.7682)$.

2. The solution in this case is $x(t) = A \cos(3t/2 + \varphi)$, and $x'(t) = -3A \sin(3t/2 + \varphi)/2$. Using the initial conditions, $x(0) = A \cos(\varphi) = 2\pi$, and $x'(0) = -3A \sin(\varphi)/2 = 3\pi$. Therefore, $A = 2\sqrt{2}\pi$, and $\varphi = -\pi/4$.

3. The solution in this case is $x(t) = A \cos(\pi t + \varphi)$, and $x'(t) = -A\pi \sin(\pi t + \varphi)$. Using the initial conditions, $x(0) = A \cos(\varphi) = 1$, and $x'(0) = -\pi A \sin(\varphi)/2 = \sqrt{3}\pi$. Therefore, $A = 2$, and $\varphi = -\pi/3$.

4. The differential equation for this problem is $4x'' + 100x = 0$, where $x(t)$ is the displacement (given in m) and t is time (measured in seconds). The auxiliary equation is then $m^2 + 25 = 0$ or $m = \pm 5i$. Therefore, the general solution is $x(t) = A \cos(5t) + B \sin(5t)$. Since $x(0) = 0$, $A = 0$. Next, $x'(t) = 5B \cos(5t)$. Because $x'(0) = 5B = 5$, $B = 1$. The final solution is $x(t) = 5 \cos(5t)$.

5. From the information provided by the original weight, we have that $Mg = kL$. Therefore, the differential equation is $m d^2x/dt^2 = -Mgx/L$. Let $\omega^2 = Mg/mL$. Then the problem can be written $x'' + \omega^2x = 0$, $x(0) = s_0$, $x'(0) = v_0$. The general solution is $x(t) = A \cos(\omega t) + B \sin(\omega t)$. From the $x(0)$ initial condition, $A = s_0$. From the $x'(0)$ initial condition, $\omega B = v_0$. Therefore, the final answer is $x(t) = s_0 \cos(\omega t) + v_0 \sin(\omega t)/\omega$, and $v(t) = v_0 \cos(\omega t) - \omega s_0 \sin(\omega t)$.

6. If $x(t)$ denotes the distance from the origin, the differential equation is $mx'' - kx = 0$. Its general solution is $x(t) = Ae^{t\sqrt{k/m}} + Be^{-t\sqrt{k/m}}$, where k is the constant of proportionality.

Because $x(0) = a$, we have that $x(0) = a = A + B$. Next, we compute $x'(t)$ or

$$x'(t) = \sqrt{k} \left(Ae^{t\sqrt{k/m}} - Be^{-t\sqrt{k/m}} \right) / \sqrt{m}.$$

Therefore, $x'(0) = a\sqrt{k} = \sqrt{k}(A - B)/\sqrt{m}$, or $A - B = a\sqrt{m}$. Solving for A and B , the final solution is

$$x(t) = \frac{a}{2} \left[(1 + \sqrt{m})e^{t\sqrt{k/m}} + (1 - \sqrt{m})e^{-t\sqrt{k/m}} \right].$$

Section 2.3

1. The differential equation is $\frac{1}{2}x'' + 3x' + 4x = 0$, which has the characteristic polynomial $m^2 + 6m + 8 = (m + 4)(m + 2) = 0$. The general solution is $x(t) = Ae^{-2t} + Be^{-4t}$. Because $x(0) = A + B = 2$, and $x'(0) = -2A - 4B = 0$, $A = -2B$ or $B = -2$ and $A = 4$. Therefore, the final answer is $x(t) = 4e^{-2t} - 2e^{-4t}$.

2. The differential equation is $x'' + 10x' + 125x = 0$, which has the characteristic polynomial $m^2 + 10m + 125 = (m + 5)^2 + 100 = 0$. The general solution is $x(t) = e^{-5t} [A \cos(10t) + B \sin(10t)]$. Because $x(0) = A = 3$ and $x'(0) = -5A + 10B = 25$ or $B = 4$, the final answer is $x(t) = e^{-5t} [3 \cos(10t) + 4 \sin(10t)]$.

3. The differential equation is $4x'' + 20x' + 169x = 0$, which has the characteristic polynomial $4m^2 + 20m + 169 = (m + \frac{1}{2})^2 + 36 = 0$. The general solution

is $x(t) = e^{-5t/2} [A \cos(6t) + B \sin(6t)]$. Because $x(0) = A = 4$ and $6B = 16 + 10 = 26$ or $B = 13/3$, the final answer is $x(t) = e^{-5t/2} [4 \cos(6t) + \frac{13}{3} \sin(6t)]$.

4. The amplitude of the oscillations decays by 50% by the time $t = \ln(2)/\lambda$. The period of one oscillation is $2\pi/\sqrt{\omega^2 - \lambda^2}$. Therefore, the minimum number of oscillations before the amplitude decays 50% is the first integer equal to or greater than $\ln(2)\sqrt{1 - (\lambda/\omega)^2}/(2\pi\lambda/\omega)$.

5. The characteristic polynomial is $m^2 + cm + 4 = 0$. The roots are equal when $c = 4$ when $m = -2$.

6. The characteristic polynomial is $m^2 + \frac{1}{4}cm + \frac{9}{4} = (m + \frac{1}{8}c)^2 + \frac{9}{4} - \frac{1}{64}c^2 = 0$. Therefore, we are overdamped when $c > 12$, underdamped when $c < 12$, and critically damped when $c = 12$.

7. For an overdamped system, $x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$, $r_{1,2} = (-c \pm \sqrt{c^2 - 4km})/(2m)$ with $c^2 > 4km$.

Let us find the value of t at which $x(t) = 0$. This occurs when $(r_1 - r_2)t - \ln(-C_2/C_1)$. If $C_2/C_1 > 0$, there is no solution. If $C_2/C_1 < 0$, there is only one solution if $r_1 > r_2$. If $r_1 < r_2$, there is no crossing.

Section 2.4

1. To find the homogeneous solution, we solve $y_H'' + 4y_H' + 3y_H = 0$. Its solution is $y_H(x) = Ae^{-3x} + Be^{-x}$. For the particular solution, we guess $y_p(x) = Cx + D$, so that $y_p'(x) = C$, and $y_p''(x) = 0$. Substitution into the differential equation yields $4C + 3Cx + 3D = x + 1$, and $C = \frac{1}{3}$ and $D = -\frac{1}{9}$. The general solution is therefore $y(x) = Ae^{-3x} + Be^{-x} + \frac{1}{3}x - \frac{1}{9}$.

2. To find the homogeneous solution, we solve $y_H'' - y_H = 0$. Its solution is $y_H(x) = Ae^x + Be^{-x}$. For the particular solution, we guess $y_p(x) = Cxe^x + De^{-2x}$, yielding $y_p'(x) = Ce^x + Cxe^x - 2De^{-2x}$, and $y_p''(x) = 2Ce^x + Cxe^x + 4De^{-2x}$. Substitution into the differential equation gives $2Ce^x + 3De^{-2x} = e^x - 2e^{-2x}$, or $C = \frac{1}{2}$ and $D = -\frac{2}{3}$. The general solution is therefore $y(x) = Ae^x + Be^{-x} + \frac{1}{2}xe^x - \frac{2}{3}e^{-2x}$.

3. To find the homogeneous solution, we solve $y_H'' + 2y_H' + 2y_H = 0$. Its solution is $y_H(x) = e^{-x}[A \cos(x) + B \sin(x)]$. For the particular solution, we guess $y_p(x) = Cx^2 + Dx + E$, yielding $y_p'(x) = 2Cx + D$, and $y_p''(x) = 2C$. Substitution into the differential equation gives $2C + 4Cx + 2D + 2Cx^2 + 2Dx + 2E = 2x^2 + 2x + 4$, or $C = 1$, $D = -1$ and $E = 2$. The general solution is therefore $y(x) = e^{-x}[A \cos(x) + B \sin(x)] + x^2 - x + 2$.

4. To find the homogeneous solution, we solve $y_H'' + y_H' = 0$. Its solution is $y_H(x) = A + Be^{-x}$. For the particular solution, we guess $y_p(x) = Cx^3 + Dx^2 +$

Ex , yielding $y'_p(x) = 3Cx^2 + 2Dx + E$ and $y''_p(x) = 6Cx + 2D$. Substitution into the differential equation gives $6Cx + 2D + 3Cx^2 + 2Dx + E = x^2 + 2x$, or $C = \frac{1}{3}$ and $D = E = 0$. The general solution is therefore $y(x) = Ax + Be^{-x} + \frac{1}{3}x^3$.

5. To find the homogeneous solution, we solve $y''_H + 2y'_H = 0$. Its solution is $y_H(x) = A + Be^{-2x}$. For the particular solution, we guess $y_p(x) = Cx^2 + Dx + Exe^{-2x}$, yielding $y'_p(x) = 2Cx + D + Ee^{-2x} - 2Exe^{-2x}$, and $y''_p(x) = 2C + 4Exe^{-2x} - 4Ee^{-2x}$. Substitution into the differential equation gives $4Cx + 2C + 2D + 2Ee^{-2x} = 2x + 5 - e^{-2x}$, or $C = \frac{1}{2}$, $D = 2$, and $E = \frac{1}{2}$. The general solution is therefore $y(x) = A + Be^{-2x} + \frac{1}{2}x^2 + 2x + \frac{1}{2}e^{-2x}$.

6. To find the homogeneous solution, we solve $y''_H - 4y'_H + 4y_H = 0$. Its solution is $y_H(x) = Ae^{2x} + Bxe^{2x}$. For the particular solution, we guess $y_p(x) = Cx^3e^{2x} + Dx^2e^{2x}$, yielding $y'_p(x) = 3Cx^2e^{2x} + 2Cx^3e^{2x} + 2Dxe^{2x} + 2Dx^2e^{2x}$, and $y''_p(x) = 6Cxe^{2x} + 12Cx^2e^{2x} + 4Cx^3e^{2x} + 2De^{2x} + 8Dxe^{2x} + 4Dx^2e^{2x}$. Substitution into the differential equation gives $6Cxe^{2x} + 2De^{2x} = xe^{2x} + e^{2x}$, or $C = \frac{1}{6}$ and $D = \frac{1}{2}$. The general solution is therefore $y(x) = Ae^{2x} + Bxe^{2x} + (\frac{1}{2}x^2 + \frac{1}{6}x^3)e^{2x}$.

7. To find the homogeneous solution, we solve $y''_H + 4y'_H + 4y_H = 0$. Its solution is $y_H(x) = (A + Bx)e^{-2x}$. For the particular solution, we guess $y_p(x) = Ce^x + Dxe^x$, yielding $y'_p(x) = Ce^x + De^x + Dxe^x$, and $y''_p(x) = Ce^x + 2De^x + Dxe^x$. Substitution into the differential equation gives $(9C + 6D)e^x + 9Dxe^x = xe^x$, or $C = -\frac{2}{27}$ and $D = \frac{1}{9}$. The general solution is therefore $y(x) = (A + Bx)e^{-2x} + (\frac{1}{9}x - \frac{2}{27})e^x$.

8. To find the homogeneous solution, we solve $y''_H - 4y_H = 0$. Its solution is $y_H(x) = A \cosh(2x) + B \sinh(2x)$. For the particular solution, we guess $y_p(x) = Cx \cosh(2x) + Dx \sinh(2x)$, yielding $y'_p(x) = C \cosh(2x) + D \sinh(2x) + 2Cx \sinh(2x) + 2Dx \cosh(2x)$, and $y''_p(x) = 4C \sinh(2x) + 4D \cosh(2x) + 4Cx \cosh(2x) + 4Dx \sinh(2x)$. Substitution into the differential equation gives $y''_p - 4y_p = 4C \sinh(2x) + 4D \cosh(2x) = 4 \sinh(2x)$, or $D = 0$ and $C = 1$. The general solution is therefore $y(x) = A \cosh(2x) + B \sinh(2x) + x \cosh(2x)$.

9. To find the homogeneous solution, we solve $y''_H + 9y_H = 0$. Its solution is $y_H(x) = A \cos(3x) + B \sin(3x)$. For the particular solution, we guess $y_p(x) = Cx^2 \cos(3x) + Dx^2 \sin(3x) + Ex \cos(3x) + Fx \sin(3x)$, yielding $y'_p(x) = (2Cx + 3Dx^2 + E + 3Fx) \cos(3x) + (2Dx - 3Cx^2 + F - 3Ex) \sin(3x)$, and $y''_p(x) = (2C - 9Cx^2 - 9Ex + 12Dx + 6F) \cos(3x) + (2D - 9Dx^2 - 12Cx - 6E - 9Fx) \sin(3x)$. Substitution into the differential equation gives $y''_p + 9y_p = (2C + 12Dx + 6F) \cos(3x) + (2D - 12Cx - 6E) \sin(3x) = x \cos(3x)$, or $2C + 6F = 0$, $2D - 6E = 0$, $-12C = 0$, and $12D = 1$. The general solution is therefore $y(x) = A \cos(3x) + B \sin(3x) + \frac{1}{12}x^2 \sin(3x) + \frac{1}{36}x \cos(3x)$.

10. To find the homogeneous solution, we solve $y_H'' + y_H = 0$. Its solution is $y_H(x) = A \cos(x) + B \sin(x)$. For the particular solution, we guess $y_p(x) = Cx^2 \cos(x) + Dx^2 \sin(x) + Ex \cos(x) + Fx \sin(x)$, yielding $y_p'(x) = (2Cx + Dx^2 + E + Fx) \cos(x) + (2Dx - Cx^2 + F - Ex) \sin(x)$, and $y_p''(x) = (2C - Cx^2 - Ex + 4Dx + 2F) \cos(x) + (2D - Dx^2 - 4Cx - 2E - Fx) \sin(x)$. Substitution into the differential equation gives $y_p'' + 9y_p = (2C + 4Dx + 2F) \cos(x) + (2D - 4Cx - 2E) \sin(x) = \sin(x) + x \cos(x)$, or $2C + 2F = 0$, $2D - 2E = 1$, $-4C = 0$, and $4D = 1$. The general solution is therefore $y(x) = A \cos(x) + B \sin(x) + \frac{1}{4} [x^2 \sin(x) - x \cos(x)]$.

11. Using the method of undetermined coefficients, the homogeneous differential equation is $y_H'' + 2ay_H = 0$. Its solution is $y_H(x) = A + Be^{-2ax}$. For the particular solution, we guess $y_p(x) = Cx + D \sin(2\omega x) + E \cos(2\omega x)$, yielding $y_p'(x) = C + 2\omega D \cos(2\omega x) - 2\omega E \sin(2\omega x)$, and $y_p''(x) = -4\omega^2 D \sin(2\omega x) - 4\omega^2 E \cos(2\omega x)$. The trick here is to recognize that the differential equation can be rewritten as $y'' + 2ay = \frac{1}{2}[1 - \cos(2\omega x)]$. The form of the particular solution follows directly. Substitution into the differential equation gives $y_p'' + 2ay_p = 2aC - 4(a\omega E + \omega^2 D) \sin(2\omega x) + 4(a\omega D - \omega^2 E) \cos(2\omega x)$. Matching up terms, we find that

$$C = \frac{1}{4a}, \quad aE = -\omega D, \quad D = -\frac{a}{8\omega(\omega^2 + a^2)} \quad \text{and} \quad E = \frac{1}{8(\omega^2 + a^2)}.$$

Therefore, the general solution is

$$y(x) = A + Be^{-2ax} + \frac{x}{4a} + \frac{\omega \cos(2\omega x) - a \sin(2\omega x)}{8\omega(a^2 + \omega^2)}.$$

To find A and B , we use the initial conditions:

$$y(0) = A + B + \frac{1}{8(a^2 + \omega^2)} = 0, \quad \text{and} \quad y'(0) = -2aB + \frac{1}{4a} - \frac{a}{4(a^2 + \omega^2)} = 0.$$

Solving for A and B , $A = -1/(8a^2)$ and $B = \omega^2/[8a^2(a^2 + \omega^2)]$. Therefore, the complete solution is

$$y(x) = \frac{2ax - 1}{8a^2} + \frac{\omega^2 e^{-2ax}}{8a^2(a^2 + \omega^2)} - \frac{a \sin(2\omega x) - \omega \cos(2\omega x)}{8\omega(a^2 + \omega^2)}.$$

Turning to the integration technique, we begin by noting that

$$\int_0^x y'' d\xi + 2a \int_0^x y' d\xi = \frac{1}{2} \int_0^x [1 - \cos(2\omega\xi)] d\xi,$$

or

$$y'(x) - y'(0) + 2a[y(x) - y(0)] = \frac{x}{2} - \frac{\sin(2ax)}{4a}.$$

Substituting the initial conditions yields

$$y' + 2ay = \frac{x}{2} - \frac{\sin(2ax)}{4a}.$$

Since this differential equation is already in canonical form, $P(x) = 2a$, and $\mu(x) = e^{2ax}$. Multiplying by the integrating factor, we have that

$$\frac{d}{dx} [e^{2ax} y(x)] = \frac{x}{2} e^{2ax} - \frac{\sin(2\omega x)}{4\omega} e^{2ax}.$$

Integrating both sides of this equation,

$$e^{2ax} y(x) - y(0) = \frac{1}{2} \int_0^x \xi e^{2a\xi} d\xi - \frac{1}{4\omega} \int_0^x \sin(2\omega\xi) e^{2a\xi} d\xi$$

or

$$e^{2ax} y(x) = \frac{e^{2a\xi}}{8a^2} (2a\xi - 1) \Big|_0^x - \frac{e^{2a\xi}}{8\omega(a^2 + \omega^2)} [a \sin(2\omega\xi) - \omega \cos(2\omega\xi)] \Big|_0^x.$$

Solving for $y(x)$,

$$y(x) = \frac{2ax - 1}{8a^2} + \frac{\omega^2 e^{-2ax}}{8a^2(a^2 + \omega^2)} - \frac{a \sin(2\omega x) - \omega \cos(2\omega x)}{8\omega(a^2 + \omega^2)}.$$

Section 2.5

1. Because the characteristic polynomial is $m^2 + 6m + 18 = (m + 3)^2 + 9 = 0$, the homogeneous solution is $y_H(t) = e^{-3t} [A \cos(3t) + B \sin(3t)]$. Therefore, resonance will occur when $\gamma = 3$.

2. Because the characteristic polynomial is $m^2 + 2m + 2 = (m + 1)^2 + 1 = 0$, the homogeneous solution is $x_H(t) = e^{-t} [A \cos(t) + B \sin(t)]$. For the particular solution we guess $x_p(t) = C \cos(2t) + D \sin(2t)$ along with $x'_p(t) = -2C \sin(2t) + 2D \cos(2t)$, and $x''_p(t) = -4C \cos(2t) - 4D \sin(2t)$. Substitution into the differential equation yields two linear equations for C and D : $-2C + 4D = 0$, $-2D - 4C = 10$, or $C = -2$ and $D = -1$. Therefore, the general solution is $x(t) = e^{-t} [A \cos(t) + B \sin(t)] - 2 \cos(2t) - \sin(2t)$ along with $x'(t) = -e^{-t} [A \cos(t) + B \sin(t)] + e^{-t} [-A \sin(t) + B \cos(t)] + 4 \sin(2t) - 2 \cos(2t)$. Substituting the initial conditions, $x'(0) = -A + B - 2 = 0$, $x(0) = A - 2 = x_0$. Solving these equations and substituting back into the general solution yields the final answer $x(t) = e^{-t} [(2 + x_0) \cos(t) + (4 + x_0) \sin(t)] - 2 \cos(2t) - \sin(2t)$.

3. If x is taken as positive in the downward directions,

$$m \frac{d^2x}{dt^2} = \sum \text{ forces} = mg - kx,$$

or

$$m \frac{d^2x}{dt^2} + kx = mg.$$

Because the system is initially at rest and the coordinate system is chosen so that $x(0) = 0$, $x'(0) = 0$.

The general solution to this differential equation is $x(t) = A \cos(\omega t) + B \sin(\omega t) + mg/k$, where $\omega^2 = k/m$. Since $x'(0) = 0$, $B = 0$. Because $x(0) = 0$, $A = -mg/k$. Therefore, $x(t) = mg[1 - \cos(\omega t)]/k$.

4. Starting with the differential equation

$$L \frac{dI}{dt} + \frac{1}{C} \int I dt = E_0[1 - \cos(\omega t)],$$

we take its derivative with respect to time and obtain

$$L \frac{d^2I}{dt^2} + \frac{I}{C} = \omega E_0 \sin(\omega t).$$

The general solution is

$$I(t) = A \sin(\omega_1 t + \theta) - \frac{\omega E_0 \sin(\omega t)}{L\omega^2 - 1/C},$$

where $\omega_1 = 1/\sqrt{LC}$. The initial condition $I(0) = 0$ yields $\theta = 0$. On the other hand, the initial condition $I'(0) = 0$ gives $A = \omega_1 \omega^2 E_0 / (L\omega^2 - 1/C)$. Therefore, the final solution is

$$I(t) = \frac{\omega_1 \omega^2 E_0}{L\omega^2 - 1/C} \sin(\omega_1 t) - \frac{\omega E_0 \sin(\omega t)}{L\omega^2 - 1/C}.$$

5. Because the characteristic polynomial is $mp^2 + cp + k = 0$ or $[p + c/(2m)]^2 + \omega_0^2 = 0$, $\omega_0^2 = k/m - c^2/(4m^2)$, the homogeneous solution is $x_H(t) = e^{-ct/(2m)} [A \cos(\omega_0 t) + B \sin(\omega_0 t)]$. The particular solution is $x_p(t) = C \sin(\omega t - \varphi) = C \cos(\varphi) \sin(\omega t) - C \sin(\varphi) \cos(\omega t)$, along with $x'_p(t) = C\omega \cos(\varphi) \cos(\omega t) + C\omega \sin(\varphi) \sin(\omega t)$, and $x''_p(t) = -C\omega^2 \cos(\varphi) \sin(\omega t) + C\omega^2 \sin(\varphi) \cos(\omega t)$. Substituting into the differential equation, we obtain the following system of linear equations: $c\omega C \sin(\varphi) + (k - m\omega^2)C \cos(\varphi) = F_0$, and $c\omega C \cos(\varphi) - (k - m\omega^2)C \sin(\varphi) = 0$. The second equation immediately yields the relationship $\tan(\varphi) = c\omega/(k - m\omega^2)$. Solving for C , we find that

$$C = F_0 / \sqrt{c^2\omega^2 + (k - m\omega^2)^2}.$$

Therefore, the general solution is

$$x(t) = e^{-ct/(2m)} [A \cos(\omega_0 t) + B \sin(\omega_0 t)] + \frac{F_0 \sin(\omega t - \varphi)}{\sqrt{c^2 \omega^2 + (k - m\omega^2)^2}}.$$

6. The characteristic polynomial is $m^2 + Rm/L + 1/(LC) = [m + R/(2L)]^2 + [1/(LC) - R^2/(4L^2)] = 0$. If $\omega^2 = 1/(LC) - R^2/(4L^2) > 0$, then $Q(t) = e^{-Rt/(2L)} [A \cos(\omega t) + B \sin(\omega t)]$. As $t \rightarrow \infty$, $Q(t) \rightarrow 0$.

If $\omega^2 = 0$, then $Q(t) = e^{-Rt/(2L)} (A + Bt)$. Again, as $t \rightarrow \infty$, $Q(t) \rightarrow 0$.

Finally, if $\omega^2 < 0$, then $Q(t) = Ae^{[\Gamma - R/(2L)]t} + Be^{[-\Gamma - R/(2L)]t}$, where $\Gamma^2 = -\omega^2$ and $\Gamma > 0$. Because $\Gamma < R/(2L)$, $Q(t) \rightarrow 0$ as $t \rightarrow \infty$.

Section 2.6

1. First we find the homogeneous solution to the differential equation $y_H'' - 4y_H' + 3y_H = 0$. Its solution is $y_H(x) = Ae^x + Be^{3x}$. According to variation of parameters, the particular solution is of the form $y_p(x) = u_1(x)e^x + u_2(x)e^{3x}$. Substitution into the differential equation yields the following system of equations $e^x u_1'(x) + e^{3x} u_2'(x) = 0$ and $e^x u_1'(x) + 3e^{3x} u_2'(x) = e^{-x}$. Solving for $u_1'(x)$ and $u_2'(x)$, $u_1'(x) = -\frac{1}{2}e^{-2x}$ or $u_1(x) = \frac{1}{4}e^{-2x}$, and $u_2'(x) = \frac{1}{2}e^{-4x}$ or $u_2(x) = -\frac{1}{8}e^{-4x}$. The general solution is $y(x) = Ae^x + Be^{3x} + u_1(x)e^x + u_2(x)e^{3x} = Ae^x + Be^{3x} + \frac{1}{8}e^{-x}$.

2. First we find the homogeneous solution to the differential equation $y_H'' - y_H' - 2y_H = 0$. Its solution is $y_H(x) = Ae^{2x} + Be^{-x}$. According to variation of parameters, the particular solution is of the form $y_p(x) = u_1(x)e^{2x} + u_2(x)e^{-x}$. Substitution into the differential equation yields the following system of equations $e^{2x} u_1'(x) + e^{-x} u_2'(x) = 0$ and $2e^{2x} u_1'(x) - e^{-x} u_2'(x) = x$. Solving for $u_1'(x)$ and $u_2'(x)$, $u_1'(x) = \frac{1}{3}xe^{-2x}$ or $u_1(x) = -\frac{1}{12}(2x + 1)e^{-2x}$, and $u_2'(x) = -\frac{1}{3}xe^x$ or $u_2(x) = -\frac{1}{3}(x - 1)e^x$. The general solution is $y(x) = Ae^{2x} + Be^{-x} + u_1(x)e^{2x} + u_2(x)e^{-x} = Ae^{2x} + Be^{-x} - \frac{1}{2}x + \frac{1}{4}$.

3. First we find the homogeneous solution to the differential equation $y_H'' - 4y_H = 0$. Its solution is $y_H(x) = Ae^{2x} + Be^{-2x}$. According to variation of parameters, the particular solution is of the form $y_p(x) = u_1(x)e^{2x} + u_2(x)e^{-2x}$. Substitution into the differential equation yields the following system of equations $e^{2x} u_1'(x) + e^{-2x} u_2'(x) = 0$ and $2e^{2x} u_1'(x) - 2e^{-2x} u_2'(x) = xe^x$. Solving for $u_1'(x)$ and $u_2'(x)$, $u_1'(x) = \frac{1}{4}xe^{-x}$ or $u_1(x) = -\frac{1}{4}(x + 1)e^{-x}$, and $u_2'(x) = -\frac{1}{4}xe^{3x}$ or $u_2(x) = -\frac{1}{36}(3x - 1)e^{3x}$. The general solution is $y(x) = Ae^{2x} + Be^{-2x} + u_1(x)e^{2x} + u_2(x)e^{-2x} = Ae^{2x} + Be^{-2x} - (3x + 2)e^x/9$.

4. First we find the homogeneous solution to the differential equation $y_H'' + 9y_H' = 0$. Its solution is $y_H(x) = A \cos(3x) + B \sin(3x)$. According to variation

of parameters, the particular solution is of the form $y_p(x) = u_1(x) \cos(3x) + u_2(x) \sin(3x)$. Substitution into the differential equation yields the following system of equations $\cos(3x)u'_1(x) + \sin(3x)u'_2(x) = 0$ and $-3\sin(3x)u'_1(x) + 3\cos(3x)u'_2(x) = 2\sec(3x)$. Solving for $u'_1(x)$ and $u'_2(x)$, $u'_2(x) = \frac{2}{3}$ or $u_2(x) = 2x/3$. Then $u'_1(x) = -\frac{2\sin(3x)}{3\cos(3x)}$ or $u_1(x) = \frac{2}{9} \ln |\cos(3x)|$. The final answer is $y(x) = A \cos(3x) + B \sin(3x) + \frac{2}{9} \ln |\cos(3x)| \cos(3x) + \frac{2}{3} x \sin(3x)$.

5. First we find the homogeneous solution to the differential equation $y''_H + 4y'_H + 4y_H = 0$. Its solution is $y_H(x) = Ae^{-2x} + Bxe^{-2x}$. According to variation of parameters, the particular solution is of the form $y_p(x) = u_1(x)e^{-2x} + u_2(x)xe^{-2x}$. Substitution into the differential equation yields the following system of equations $e^{-2x}u'_1(x) + xe^{-2x}u'_2(x) = 0$ and $-2e^{-2x}u'_1(x) + (1 - 2x)e^{-2x}u'_2(x) = xe^{-2x}$. Solving for $u'_1(x)$ and $u'_2(x)$, $u'_1(x) = -x^2$, or $u_1(x) = -\frac{1}{3}x^3$, and $u'_2(x) = x$ or $u_2(x) = \frac{1}{2}x^2$. The general solution is $y(x) = (A + Bx)e^{-2x} + u_1(x)e^{-2x} + u_2(x)xe^{-2x} = (A + Bx)e^{-2x} + \frac{1}{6}x^3e^{-2x}$.

6. First we find the homogeneous solution to the differential equation $y''_H + 2ay'_H = 0$. Its solution is $y_H(x) = A' + Bxe^{-2ax}$. According to variation of parameters, the particular solution is of the form $y_p(x) = u_1(x) + u_2(x)e^{-2ax}$. Substitution into the differential equation yields the following system of equations $u'_1(x) + e^{-2ax}u'_2(x) = 0$, and $-2ae^{-2ax}u'_2(x) = \sin^2(\omega x)$. Solving for $u'_1(x)$ and $u'_2(x)$,

$$u'_2(x) = -\frac{e^{2ax}}{4a} [1 - \cos(2\omega x)],$$

or

$$u_2(x) = -\frac{e^{2ax}}{8a^2} + \frac{e^{2ax}}{8a(a^2 + \omega^2)} [a \cos(2\omega x) + \omega \sin(2\omega x)].$$

Then

$$u'_1(x) = \frac{1 - \cos(2\omega x)}{4a}, \quad \text{or} \quad u_1(x) = \frac{x}{4a} - \frac{\sin(2\omega x)}{4a\omega}.$$

Therefore,

$$\begin{aligned} y(x) &= A' + Bxe^{-2ax} + \frac{x}{4a} - \frac{\sin(2\omega x)}{8a\omega} - \frac{1}{8a^2} + \frac{a \cos(2\omega x) + \omega \sin(2\omega x)}{8a(a^2 + \omega^2)} \\ &= A + Bxe^{-2ax} + \frac{x}{4a} - \frac{a \sin(2\omega x) - \omega \cos(2\omega x)}{8\omega(a^2 + \omega^2)}. \end{aligned}$$

7. First we find the homogeneous solution to the differential equation $y''_H - 4y'_H + 4y_H = 0$. Its solution is $y_H(x) = Ae^{2x} + Bxe^{2x}$. According to variation of parameters, the particular solution is of the form $y_p(x) = u_1(x)e^{2x} + u_2(x)xe^{2x}$. Substitution into the differential equation yields the following system of equations $e^{2x}u'_1(x) + xe^{2x}u'_2(x) = 0$, and $2e^{2x}u'_1(x) + (2x+1)e^{2x}u'_2(x) =$

$(x+1)e^{2x}$. Solving for $u'_1(x)$ and $u'_2(x)$, $u'_1(x) = -x - x^2$ or $u_1(x) = -\frac{1}{2}x^2 - \frac{1}{3}x^3$, and $u'_2(x) = 1 + x$ or $u_2(x) = x + \frac{1}{2}x^2$. The general solution is $y(x) = Ae^{2x} + Bxe^{2x} + u_1(x)e^{2x} + u_2(x)xe^{2x} = Ae^{2x} + Bxe^{2x} + (\frac{1}{2}x^2 + \frac{1}{6}x^3)e^{2x}$.

8. First we find the homogeneous solution to the differential equation $y''_H - 4y_H = 0$. Its solution is $y_H(x) = Ae^x + Be^{-x}$. According to variation of parameters, the particular solution is of the form $y_p(x) = u_1(x)e^x + u_2(x)e^{-x}$. Substitution into the differential equation yields the following system of equations $e^xu'_1(x) + e^{-x}u'_2(x) = 0$, and $e^xu'_1(x) - e^{-x}u'_2(x) = \sin^2(x)$. Solving for $u'_1(x)$ and $u'_2(x)$, $u'_1(x) = \frac{1}{2}e^{-x}\sin^2(x)$, or $u_1(x) = -\frac{1}{10}e^{-x}[2 + \sin(x) + 2\cos(x)]$, and $u'_2(x) = -\frac{1}{2}e^x\sin^2(x)$, or $u_2(x) = -\frac{1}{10}e^x[2 + \sin(x) - 2\cos(x)]$. The general solution is $y(x) = Ae^x + Be^{-x} + u_1(x)e^x + u_2(x)e^{-x} = Ae^x + Be^{-x} - \frac{1}{5}[2 + \sin^2(x)]$.

9. First we find the homogeneous solution to the differential equation $y''_H - 2y'_H + y_H = 0$. Its solution is $y_H(x) = Ae^x + Cxe^x$. According to variation of parameters, the particular solution is of the form $y_p(x) = u_1(x)e^x + u_2(x)xe^x$. Substitution into the differential equation yields the following system of equations $e^xu'_1(x) + xe^xu'_2(x) = 0$ and $e^xu'_1(x) + (e^x + xe^x)u'_2(x) = e^x/x$. Solving for $u'_1(x)$ and $u'_2(x)$, $u'_1(x) = -1$, or $u_1(x) = -x$, and $u'_2(x) = 1/x$, or $u_2(x) = \ln(x)$. The general solution is $y(x) = Ae^x + Cxe^x + u_1(x)e^x + u_2(x)xe^x = Ae^x + Bxe^x + x\ln(x)e^x$.

10. First we find the homogeneous solution to the differential equation $y''_H + y_H = 0$. Its solution is $y_H(x) = A\cos(x) + B\sin(x)$. According to variation of parameters, the particular solution is of the form $y_p(x) = u_1(x)\cos(x) + u_2(x)\sin(x)$. Substitution into the differential equation yields the following system of equations $\cos(x)u'_1(x) + \sin(x)u'_2(x) = 0$, and $-\sin(x)u'_1(x) + \cos(x)u'_2(x) = \tan(x)$. Solving for $u'_1(x)$ and $u'_2(x)$, $u'_1(x) = -\sin^2(x)/\cos(x)$, or $u_1(x) = -\ln|\sec(x) + \tan(x)| + \sin(x)$, and $u'_2(x) = \sin(x)$, or $u_2(x) = -\cos(x)$. The general solution is

$$\begin{aligned} y(x) &= A\cos(x) + B\sin(x) + u_1(x)\cos(x) + u_2(x)\sin(x) \\ &= A\cos(x) + B\sin(x) - \ln\left|\frac{1+\sin(x)}{\cos(x)}\right|\cos(x) \\ &= A\cos(x) + B\sin(x) + \ln\left|\frac{1-\sin(x)}{\cos(x)}\right|\cos(x). \end{aligned}$$

Section 2.7

1. The auxiliary equation is $m(m-1)+m-1 = (m-1)(m+1) = 0$. Therefore, $y(x) = C_1x + C_2x^{-1}$.

2. The auxiliary equation is $m(m-1) + 2m - 2 = (m-1)(m+2) = 0$. Therefore, $y(x) = C_1x + C_2x^{-2}$.

3. The auxiliary equation is $m(m-1) - 2 = (m-2)(m+1) = 0$. Therefore, $y(x) = C_1x^2 + C_2x^{-1}$.
4. The auxiliary equation is $m(m-1) - m + 1 = (m-1)^2 = 0$. Therefore, $y(x) = C_1x + C_2x \ln(x)$.
5. The auxiliary equation is $m(m-1) + 3m + 1 = (m+1)^2 = 0$. Therefore, $y(x) = C_1x^{-1} + C_2x^{-1} \ln(x)$.
6. The auxiliary equation is $m(m-1) - 3m + 4 = (m-2)^2 = 0$. Therefore, $y(x) = C_1x^2 + C_2x^2 \ln(x)$.
7. The auxiliary equation is $m(m-1) - m + 5 = (m-1)^2 + 4 = 0$ so that $m = -1 \pm 2i$. Therefore, $y(x) = C_1x \cos[2 \ln(x)] + C_2x \sin[2 \ln(x)]$.
8. The auxiliary equation is $4m(m-1) + 8m + 5 = (2m+1)^2 + 4 = 0$ so that $m = -\frac{1}{2} \pm i$. Therefore, $y(x) = C_1x^{-1/2} \cos[\ln(x)] + C_2x^{-1/2} \sin[\ln(x)]$.
9. The auxiliary equation is $m(m-1) + m + 1 = m^2 + 1 = 0$ so that $m = \pm i$. Therefore, $y(x) = C_1 \cos[\ln(x)] + C_2 \sin[\ln(x)]$.
10. The auxiliary equation is $m(m-1) - 3m + 13 = (m-2)^2 + 9 = 0$ so that $m = 2 \pm 3i$. Therefore, $y(x) = C_1x^2 \cos[3 \ln(x)] + C_2x^2 \sin[3 \ln(x)]$.
11. The auxiliary equation is $m(m-1)(m-2) - 2m(m-1) - 2m + 8 = m^3 - 5m^2 + 2m + 8 = (m-2)(m-4)(m+1) = 0$. Therefore, $y(x) = C_1x^2 + C_2x^4 + C_3x^{-1}$.
12. Let $t = \ln(x)$ and $y(x) = Y(t)$, then $Y'' - 3Y' - 4Y = e^t$. The homogeneous solution is $Y_H(t) = Ae^{4t} + Be^{-t}$. From the method of undetermined coefficients, $Y_p(t) = -\frac{1}{6}e^t$. Transforming back, $y(x) = Ax^4 + B/x - x/6$.

Section 2.8

1. The solution to the differential equation is $x(t) = C_1e^t + C_2e^{2t}$. Therefore, any point on the phase diagram except for $x = v = 0$ has a trajectory off to infinity.
2. Multiplying the differential equation by x' and setting $v = x'$ on the left side, $vv' = x^3x' - xx'$. Integration yields $\frac{1}{2}v^2 - \frac{1}{4}x^4 + \frac{1}{2}x^2 = C$. If $C = 0$, $x = v = 0$ and we have an equilibrium point where the energy is at a minimum. If $C = \frac{1}{4}$, we have another equilibrium point corresponding to $x = \pm 1$ and $v = 0$. Note that $C \leq \frac{1}{4}$.
3. The curves on the phase diagram are given by $v = 2x + C$. The equilibrium points are $v = 0$ for all x . Therefore the entire abscissa contain all of

the equilibria. Any displacement from this equilibrium moves off to infinity. Therefore, the equilibrium is unstable.

4. The curves on the phase diagram are given by $v^2 + \operatorname{sgn}(x)x^2 = C$. The equilibrium points are $v = 0$ and $v' = -\operatorname{sgn}(x)x = 0$. Therefore, the point $(0, 0)$ is the equilibrium point. Any displacement from this equilibrium moves off to infinity. Therefore, the equilibrium is unstable.

5. The curves on the phase diagram are given by $v^2 - x = C$ if $|x| > 2$ and $v = C$ if $|x| < 2$. The equilibrium points are $v = 0$ and $|x| < 2$. Therefore, there are infinite number of equilibrium points, all located along the x -axis for $|x| < 2$. Any displacement from these equilibria moves off to infinity and all of the equilibria are unstable.

Section 2.9

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%                               MATLAB Code for Pendulum Problem
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% initialize parameters

clear; b = 0.22; g = 9.8; k = 0.02; L = 1; omega_0_sq = g/L;
omega_sq = omega_0_sq - b*b/4; omega = sqrt(omega_sq);
delta_t = 0.005; t = [0:delta_t:50]; % set up the time
theta(1) = 10*pi/180; thetap(1) = 0; % initial conditions

% time step to find position and velocity
%   of the pendulum at later times

for i = 1:length(t)-1
    theta_g = theta(i) + delta_t*thetap(i);
    thetap_g = thetap(i) - b*delta_t*thetap(i) ...
               - omega_0_sq*delta_t*theta(i);
    avg_1 = (theta(i) + theta_g) / 2;
    avg_2 = (thetap(i) + thetap_g) / 2;
    theta(i+1) = theta(i) + delta_t*avg_2;
    thetap(i+1) = thetap(i) - b*delta_t*avg_2 ...
                  - omega_0_sq*delta_t*avg_1;
    if ((abs(theta(i)) < delta_t/2) & (thetap(i) > 0))
        thetap(i+1) = thetap(i+1) + k; end % impulse forcing
end
```

```
% plot the results

plot(theta,thetap,'k'); xlabel('\theta','FontSize',25);
ylabel('\theta^{\prime}','FontSize',25)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Section 3.1

1.

$$A + B = \begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix} = B + A$$

2.

$$A - B = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}$$

$$B - A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 1 & 0 \end{pmatrix}$$

3.

$$3A - 2B = \begin{pmatrix} 9 & 12 \\ 3 & 6 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ -1 & 2 \end{pmatrix}$$

$$3(2A - B) = 3 \left[\begin{pmatrix} 6 & 8 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \right] = 3 \begin{pmatrix} 5 & 7 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 15 & 21 \\ 0 & 6 \end{pmatrix}$$

4.

$$A^T = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}, \quad B^T = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad (B^T)^T = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

5.

$$(A + B)^T = \begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix}$$

$$A^T + B^T = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix}$$

6.

$$B + B^T = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$$

$$B - B^T = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$