Finite Element Method for Boundary Value Problems Mathematics and Computations 1st edition Surana Solutions Manual

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Chapter 2

Solutions of Problems

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Prob. 2.1

$$\frac{d\phi}{dx} - \frac{1}{Pe}\frac{d^2\phi}{dx^2} = 0 \quad \forall \; x \in \Omega = (0,1) \subset R^1 \qquad \text{——— (1)}$$

$$\phi(0) = 1, \quad \phi(1) = 0 \qquad \text{——— (2)}$$

Solution:

a) Differential operator in (1)

$$A = \frac{d}{dx} - \frac{1}{Pe}\frac{d^2}{dx^2} \qquad \text{——— (3)}$$

∴ (1) Can be written as

$$A\phi = 0 \; ; \; x \in \Omega = (0,1) \subset R^1 \qquad \text{——— (4)}$$

b) Linearity of A:

$\forall\, u, v \in D_A$, domain of definition of A

and $\forall\, \alpha, \beta \in R$

If we can show that

$$A(\alpha u + \beta v) = \alpha Au + \beta Av \quad \text{Then A is linear} \qquad \text{— (5)}$$

Consider $A(\alpha u + \beta v)$

$$A(\alpha u + \beta v) = \left(\frac{d}{dx} - \frac{1}{Pe}\frac{d^2}{dx^2}\right)(\alpha u + \beta v)$$

$$= \left(\frac{d}{dx} - \frac{1}{Pe}\frac{d^2}{dx^2}\right)(\alpha u) + \left(\frac{d}{dx} - \frac{1}{Pe}\frac{d^2}{dx^2}\right)(\beta v)$$

using product rule of differentiation for αu and βv.

$$A(\alpha u + \beta v) = \alpha\left(\frac{d}{dx} - \frac{1}{Pe}\frac{d^2}{dx^2}\right)u + \beta\left(\frac{d}{dx} - \frac{1}{Pe}\frac{d^2}{dx^2}\right)v$$

$$= \alpha Au + \beta Av \qquad \text{——— (6)}$$

Hence in this case

$$A(\alpha u + \beta v) = \alpha Au + \beta Av \text{ holds.}$$

∴ The differential operator A is linear.

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Prob 2.2

$$\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) = 0 \quad \forall\, x \in \Omega = (0,1) \subset R^1 \qquad \text{——— (1)}$$

$$\phi(0) = 0 \qquad \left(a(x)\frac{d\phi}{dx}\right)\Big|_1 = P_1 \qquad \text{——— (2)}$$

Solution :

(a) differential operator in (1)

$$A = \frac{d}{dx}\left(a(x)\frac{d}{dx}\right) \qquad \text{——— (3)}$$

$\therefore$ (1) can be written as

$$A\phi = 0, \forall\, x \in \Omega = (0,1) \subset R^1 \qquad \text{——— (4)}$$

(b) Linearity of A :

$\forall\, u, v \in D_A$

and $\forall\, \alpha, \beta \in R$

If we can show that

$A(\alpha u + \beta v) = \alpha Au + \beta Av$ then A is linear ——— (5)

Consider $A(\alpha u + \beta v)$

$$A(\alpha u + \beta v) = \frac{d}{dx}\left(a(x)\frac{d}{dx}\right)(\alpha u + \beta v)$$

$$= \left(\frac{d}{dx}\left(a(x)\frac{d}{dx}\right)\right)(\alpha u) + \left(\frac{d}{dx}\left(a(x)\frac{d}{dx}\right)\right)(\beta v)$$

using product rule of differentiation for αu and βv

$$A(\alpha u + \beta v) = \alpha\left(\frac{d}{dx}\left(a(x)\frac{d}{dx}\right)\right)u + \beta\left(\frac{d}{dx}\left(a(x)\frac{d}{dx}\right)\right)v$$

$$= \alpha Au + \beta v \qquad \text{——— (6)}$$

Hence, in this case

$A(\alpha u + \beta v) = \alpha Au + \beta Av$ holds

$\therefore$ The differential operator A is linear

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Prob. 2.3

$$\phi \frac{d\phi}{dx} - \frac{1}{Re} \frac{d^2\phi}{dx^2} = 0 \quad \forall\, x \in \Omega = (0,1) \subset R^1 \qquad (1)$$

$$\phi(0) = 1 \;; \quad \phi(1) = 0 \qquad (2)$$

Solution:

(a) differential operator A:

$$A = \phi \frac{d}{dx} - \frac{1}{Re} \frac{d^2}{dx^2} \qquad (3)$$

∴ (1) can be written as

$$A\phi = 0 \quad \forall\, x \in \Omega = (0,1) \subset R^1 \qquad (4)$$

(b) linearity of A:

$\forall\, u, v \in D_A$

and $\forall\, \alpha, \beta \in R$

If we can show that

$$A(\alpha u + \beta v) = \alpha Au + \beta Av \text{ then } A \text{ is linear} \qquad (5)$$

Consider $A(\alpha u + \beta v)$. ; substitute $\alpha u + \beta v$ for ϕ in (4) and (3)

$$A(\alpha u + \beta v) = \left((\alpha u + \beta v)\frac{d}{dx} - \frac{1}{Re}\frac{d^2}{dx^2}\right)(\alpha u + \beta v)$$

$$= \alpha(\alpha u + \beta v)\frac{du}{dx} + \beta(\alpha u + \beta v)\frac{dv}{dx} - \frac{\alpha}{Re}\frac{d^2u}{dx^2} - \frac{\beta}{Re}\frac{d^2v}{dx^2} \qquad (6)$$

on the other hand (right side of (5))

$$\alpha Au + \beta Av = \alpha\left(u\frac{d}{dx} - \frac{1}{Re}\frac{d^2}{dx^2}\right)u + \beta\left(v\frac{d}{dx} - \frac{1}{Re}\frac{d^2}{dx^2}\right)v$$

$$= \alpha u\frac{du}{dx} + \beta v\frac{dv}{dx} - \frac{\alpha}{Re}\frac{d^2u}{dx^2} - \frac{\beta}{Re}\frac{d^2v}{dx^2} \qquad (7)$$

Even though the last two terms in (6) and (7) are exactly same, but the first two are not, hence

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$A(\alpha u + \beta v) = \alpha A u + \beta A v$ does not hold in this case. Thus, the differential operator A in (3) is not linear.

Prob. 2.4

$$B(\phi, v) = \int_1^2 a(x) \frac{d\phi}{dx} \frac{dv}{dx} dx + b(2)\phi(2)v(2) \quad \text{—(1)}$$

(a) Bilinearity of $B(\phi, v)$

If $B(\phi, v)$ is bilinear, then it must be linear in each one of its arguments i.e linear in ϕ as well as linear in v.

I. Linearity in ϕ

Let $u, w \in D_B$

and $\alpha, \beta \in R$

Then we must show that

$$B(\alpha u + \beta w, v) = \alpha B(u, v) + \beta B(w, v) \quad \text{holds} \quad \text{—(2)}$$

if $B(\phi, v)$ is linear in ϕ.

Consider

$$B(\alpha u + \beta w, v) = \int_1^2 a(x) \frac{d(\alpha u + \beta w)}{dx} \frac{dv}{dx} dx$$

$$+ b(2)(\alpha u(2) + \beta w(2))v(2) \quad \text{—(3)}$$

Expanding right side and regrouping the terms.

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$$B(\alpha u+\beta w, v) = \alpha\left(\int_1^2 a(x)\frac{du}{dx}\frac{dv}{dx}dx + b(2)u(2)v(2)\right) +$$

$$\beta\left(\int_1^2 a(x)\frac{dw}{dx}\frac{dv}{dx}dx + b(2)w(2)v(2)\right)$$

$$= \alpha B(u,v) + \beta B(w,v) \quad \text{——— (4)}$$

Hence, (2) holds, thus $B(\phi, v)$ is linear in ϕ.

II. linearity in v

Let $v_1, v_2 \in D_B$

and $\alpha_1, \alpha_2 \in R$

Then we must show that

$B(\phi, \alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 B(\phi, v_1) + \alpha_2 B(\phi, v_2)$ holds —(5)

if $B(\phi, v)$ is linear in v.

Consider

$$B(\phi, \alpha_1 v_1 + \alpha_2 v_2) = \int_1^2 a(x)\frac{d\phi}{dx}\frac{d}{dx}(\alpha_1 v_1 + \alpha_2 v_2)dx + b(2)\phi(2)(\alpha_1 v_1(2) + \alpha_2 v_2(2)) \quad \text{——— (6)}$$

Expanding the right side of (6) and rearranging the terms

$$B(\phi, \alpha_1 v_1 + \alpha_2 v_2) = \alpha_1\left(\int_1^2 a(x)\frac{d\phi}{dx}\frac{dv_1}{dx}dx + b(2)\phi(2)v_1(2)\right) +$$

$$\alpha_2\left(\int_1^2 a(x)\frac{d\phi}{dx}\frac{dv_2}{dx}dx + b(2)\phi(2)v_2(2)\right)$$

$$= \alpha_1 B(\phi, v_1) + \alpha_2 B(\phi, v_2) \quad \text{——— (7)}$$

Hence, (5) holds, thus $B(\phi, v)$ is linear in v.

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Thus, $B(\phi, v)$ is linear in ϕ as well v, hence

$B(\phi, v)$ is bilinear.

(b) Symmetry of $B(\phi, v)$

If $B(\phi, v)$ is symmetric then

$$B(\phi, v) = B(v, \phi) \quad \text{must hold.} \qquad \text{—— (8)}$$

$$B(\phi, v) = \int_1^2 a(x) \frac{d\phi}{dx} \frac{dv}{dx} dx + b(2)\phi(2)v(2) \qquad \text{—— (9)}$$

$$B(v, \phi) = \int_1^2 a(x) \frac{dv}{dx} \frac{d\phi}{dx} dx + b(2)v(2)\phi(2) \qquad \text{—— (10)}$$

$$= \int_1^2 a(x) \frac{d\phi}{dx} \frac{dv}{dx} dx + b(2)\phi(2)v(2) \qquad \text{—— (11)}$$

Comparing (9) and (11) it is straight forward to conclude that

$B(\phi, v) = B(v, \phi)$ holds.

Hence $B(\phi, v)$ is symmetric.

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Prob. 2.5

$$B(\phi, v) = \int_{-1}^{1} a(x) \frac{d^2\phi}{dx^2} \frac{d^2 v}{dx^2} dx + \int_{-1}^{1} c(x) \phi v \, dx \quad \text{—(1)}$$

(a) Bilinearity of $B(\phi, v)$

If $B(\phi, v)$ is bilinear, then it must be linear in each one of its arguments, i.e linear in ϕ as well as linear in v.

I. Linearity in ϕ

Let $\phi_1, \phi_2 \in D_B$

and $\alpha_1, \alpha_2 \in R$

Then we must show that

$B(\alpha_1\phi_1 + \alpha_2\phi_2, v) = \alpha_1 B(\phi_1, v) + \alpha_2 B(\phi_2, v)$ holds —(2)

if $B(\phi, v)$ is linear in ϕ.

Consider

$$B(\alpha_1\phi_1 + \alpha_2\phi_2, v) = \int_{-1}^{1} a(x) \frac{d^2(\alpha_1\phi_1 + \alpha_2\phi_2)}{dx^2} \frac{d^2 v}{dx^2} dx + \int_{-1}^{1} c(x)(\alpha_1\phi_1 + \alpha_2\phi_2) v \, dx \quad \text{——— (3)}$$

Expanding right side and rearranging the terms

$$B(\alpha_1\phi_1 + \alpha_2\phi_2, v) = \alpha_1 \left(\int_{-1}^{1} a(x) \frac{d^2\phi_1}{dx^2} \frac{d^2 v}{dx^2} dx + \int_{-1}^{1} c(x)\phi_1 v \, dx \right) + \alpha_2 \left(\int_{-1}^{1} a(x) \frac{d^2\phi_2}{dx^2} \frac{d^2 v}{dx^2} dx + \int_{-1}^{1} c(x)\phi_2 v \, dx \right)$$

$$= \alpha_1 B(\phi_1, v) + \alpha_2 B(\phi_2, v) \quad \text{——— (4)}$$

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Hence, (2) holds, thus $B(\phi, v)$ is linear in ϕ.

II Linearity in v

Let $v_1, v_2 \in D_B$

and $\beta_1, \beta_2 \in R$

Then we must show that

$B(\phi, \beta_1 v_1 + \beta_2 v_2) = \beta_1 B(\phi, v_1) + \beta_2 B(\phi, v_2)$ holds — (5)

if $B(\phi, v)$ is linear in v.

Consider

$$B(\phi, \beta_1 v_1 + \beta_2 v_2) = \int_{-1}^{1} a(x) \frac{d^2\phi}{dx^2} \frac{d^2(\beta_1 v_1 + \beta_2 v_2)}{dx^2} dx + \int_{-1}^{1} c(x) \phi (\beta_1 v_1 + \beta_2 v_2) dx \quad \text{—— (6)}$$

Expanding the right side of (6) and rearranging the terms.

$$B(\phi, \beta_1 v_1 + \beta_2 v_2) = \beta_1 \left(\int_{-1}^{1} a(x) \frac{d^2\phi}{dx^2} \frac{d^2 v_1}{dx^2} dx + \int_{-1}^{1} c(x) \phi v_1 dx \right) + \beta_2 \left(\int_{-1}^{1} a(x) \frac{d^2\phi}{dx^2} \frac{d^2 v_2}{dx^2} dx + \int_{-1}^{1} c(x) \phi v_2 dx \right)$$

$$= \beta_1 B(\phi, v_1) + \beta_2 B(\phi, v_2) \quad \text{—— (7)}$$

Hence, (5) holds, thus $B(\phi, v)$ is linear in v.

Thus, $B(\phi, v)$ is linear in ϕ as well as v, hence $B(\phi, v)$ is bilinear.

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(b) Symmetry of $B(\phi, v)$

If $B(\phi, v)$ is symmetric, then

$B(\phi, v) = B(v, \phi)$ must hold. —— (8)

$$B(\phi, v) = \int_{-1}^{1} a(x) \frac{d^2\phi}{dx^2} \frac{d^2v}{dx^2} dx + \int_{-1}^{1} c(x) \phi v \, dx \quad —— (9)$$

$$B(v, \phi) = \int_{-1}^{1} a(x) \frac{d^2v}{dx^2} \frac{d^2\phi}{dx^2} dx + \int_{-1}^{1} c(x) v \phi \, dx \quad —— (10)$$

$$= \int_{-1}^{1} a(x) \frac{d^2\phi}{dx^2} \frac{d^2v}{dx^2} dx + \int_{-1}^{1} c(x) \phi v \, dx \quad —— (11)$$

Comparing (9) and (11) we conclude that:

$B(\phi, v) = B(v, \phi)$ holds

Hence $B(\phi, v)$ is Symmetric.

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Prob. 2.6

$$B(\phi, v) = \int_{-1}^{1} \phi \frac{d\phi}{dx} \frac{dv}{dx} dx \qquad \text{——— (1)}$$

(a) Bilinearity of $B(\phi, v)$

If $B(\phi, v)$ is bilinear, then it must be linear in each one of its arguments i.e linear in ϕ as well as linear in v.

I. Linearity in ϕ

Let $\phi_1, \phi_2 \in D_B$

and $\alpha_1, \alpha_2 \in R$

Then we must show that

$$B(\alpha_1\phi_1 + \alpha_2\phi_2, v) = \alpha_1 B(\phi_1, v) + \beta_2 B(\phi_2, v) \text{ holds} \qquad \text{— (2)}$$

if $B(\phi, v)$ is linear in ϕ.

Consider

$$B(\alpha_1\phi_1 + \alpha_2\phi_2, v) = \int_{-1}^{1} (\alpha_1\phi_1 + \alpha_2\phi_2) \frac{d}{dx}(\alpha_1\phi_1 + \alpha_2\phi_2) \frac{dv}{dx} dx \qquad \text{— (3)}$$

$$= \alpha_1 \int_{-1}^{1} \phi_1 \frac{d}{dx}(\alpha_1\phi_1 + \alpha_2\phi_2) \frac{dv}{dx} dx +$$

$$\alpha_2 \int_{-1}^{1} \phi_2 \frac{d}{dx}(\alpha_1\phi_1 + \alpha_2\phi_2) \frac{dv}{dx} dx$$

$$= \alpha_1^2 \int_{-1}^{1} \phi_1 \frac{d\phi_1}{dx} \frac{dv}{dx} dx + \alpha_1\alpha_2 \int_{-1}^{1} \phi_1 \frac{d\phi_2}{dx} \frac{dv}{dx} dx +$$

$$\alpha_1\alpha_2 \int_{-1}^{1} \phi_2 \frac{d\phi_1}{dx} \frac{dv}{dx} dx + \alpha_2^2 \int_{-1}^{1} \phi_2 \frac{d\phi_2}{dx} \frac{dv}{dx} dx \qquad \text{— (4)}$$

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Consider

$$\alpha_1 B(\phi_1, v) = \alpha_1 \int_{-1}^{1} \phi_1 \frac{d\phi_1}{dx} \frac{dv}{dx} dx \qquad \text{——— (5)}$$

and $$\alpha_2 B(\phi_2, v) = \alpha_2 \int_{-1}^{1} \phi_2 \frac{d\phi_2}{dx} \frac{dv}{dx} dx \qquad \text{——— (6)}$$

From (4), (5) and (6) we note that (2) does not hold, thus $B(\phi, v)$ is not linear in ϕ.

II. Linearity in v

Let $v_1, v_2 \in D_B$

and $\beta_1, \beta_2 \in R$

Then, we must show that

$$B(\phi, \beta_1 v_1 + \beta_2 v_2) = \beta_1 B(\phi, v_1) + \beta_2 B(\phi, v_2) \quad \text{holds} \qquad \text{——— (7)}$$

if $B(\phi, v)$ is linear in v.

Consider

$$B(\phi, \beta_1 v_1 + \beta_2 v_2) = \int_{-1}^{1} \phi \frac{d\phi}{dx} \frac{d}{dx}(\beta_1 v_1 + \beta_2 v_2) dx$$

$$= \beta_1 \int_{-1}^{1} \phi \frac{d\phi}{dx} \frac{dv_1}{dx} dx + \beta_2 \int_{-1}^{1} \phi \frac{d\phi}{dx} \frac{dv_2}{dx} dx$$

$$= \beta_1 B(\phi, v_1) + \beta_2 B(\phi, v_2) \qquad \text{——— (8)}$$

Hence, (7) holds, thus $B(\phi, v)$ is linear in v.

From I. and II. we note that $B(\phi, v)$ is linear in v but not linear in ϕ, hence $B(\phi, v)$ is not bilinear.

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(b) Symmetry of $B(\phi, v)$

If $B(\phi, v)$ is symmetric then

$$B(\phi, v) = B(v, \phi) \text{ must hold.} \quad \text{—— (9)}$$

- Since $B(\phi, v)$ is not bilinear, it can not be symmetric, but we consider details in the following.

$$B(\phi, v) = \int_{-1}^{1} \phi \frac{d\phi}{dx} \frac{dv}{dx} dx \quad \text{—— (10)}$$

$$B(v, \phi) = \int_{-1}^{1} v \frac{dv}{dx} \frac{d\phi}{dx} dx \quad \text{—— (11)}$$

Clearly (10) and (11) are not the same, Thus $B(\phi, v)$ is not symmetric

———— o ————

Prob. 2.7

$$B(\phi, v) = \int_{-1}^{1} \frac{d\phi}{dx} v\, dx + \frac{1}{Pe} \int_{-1}^{1} \frac{d\phi}{dx} \frac{dv}{dx} dx \quad \text{—— (1)}$$

(a) Bilinearity of $B(\phi, v)$

If $B(\phi, v)$ is bilinear, then it must be linear in each one of its arguments i.e linear in ϕ as well as linear in v.

I. Linearity in ϕ

Let $\phi_1, \phi_2 \in D_B$

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and $\alpha_1, \alpha_2 \in R$

Then we must show that

$$B(\alpha_1\phi_1+\alpha_2\phi_2, v) = \alpha_1 B(\phi_1, v) + \alpha_2 B(\phi_2, v) \text{ holds} \quad \text{---} (2)$$

if $B(\phi, v)$ is linear in ϕ.

Consider

$$B(\alpha_1\phi_1+\alpha_2\phi_2, v) = \int_{-1}^{1} \frac{d}{dx}(\alpha_1\phi_1+\alpha_2\phi_2)\, v\, dx + \frac{1}{Pe}\int_{-1}^{1} \frac{d}{dx}(\alpha_1\phi_1+\alpha_2\phi_2)\frac{dv}{dx}\, dx$$

$$= \alpha_1\left(\int_{-1}^{1}\frac{d\phi_1}{dx}\, v\, dx + \frac{1}{Pe}\int_{-1}^{1}\frac{d\phi_1}{dx}\frac{dv}{dx}\, dx\right) + \alpha_2\left(\int_{-1}^{1}\frac{d\phi_2}{dx}\, v\, dx + \frac{1}{Pe}\int_{-1}^{1}\frac{d\phi_2}{dx}\frac{dv}{dx}\, dx\right)$$

$$= \alpha_1 B(\phi_1, v) + \alpha_2 B(\phi_2, v) \quad \text{---} (3)$$

Hence (2) holds, thus $B(\phi, v)$ is linear in ϕ.

(b) Linearity in v.

Let $v_1, v_2 \in D_B$

and $\beta_1, \beta_2 \in R$

Then we must show that

$$B(\phi, \beta_1 v_1 + \beta_2 v_2) = \beta_1 B(\phi, v_1) + \beta_2 B(\phi, v_2) \quad \text{---} (4)$$

must hold if $B(\phi, v)$ is linear in v.

Consider

$$B(\phi, \beta_1 v_1 + \beta_2 v_2) = \int_{-1}^{1} \frac{d\phi}{dx}(\beta_1 v_1 + \beta_2 v_2)\, dx + \frac{1}{Re}\int_{-1}^{1} \frac{d\phi}{dx}\frac{d}{dx}(\beta_1 v_1 + \beta_2 v_2)\, dx \quad \text{---} (5)$$

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or

$$B(\phi, \beta_1 v_1 + \beta_2 v_2) = \beta_1 \left(\int_{-1}^{1} \frac{d\phi}{dx} v_1 \, dx + \frac{1}{Pe} \int_{-1}^{1} \frac{d\phi}{dx} \frac{dv_1}{dx} dx \right) +$$

$$\beta_2 \left(\int_{-1}^{1} \frac{d\phi}{dx} v_2 \, dx + \frac{1}{Pe} \int_{-1}^{1} \frac{d\phi}{dx} \frac{dv_2}{dx} dx \right)$$

$$= \beta_1 B(\phi, v_1) + \beta_2 B(\phi, v_2) \quad \text{—— (6)}$$

Hence, (4) holds, thus $B(\phi, v)$ is linear in v.

Thus, $B(\phi, v)$ is linear in ϕ as well as in v, hence $B(\phi, v)$ is bilinear.

(b) Symmetry of $B(\phi, v)$

If $B(\phi, v)$ is symmetric, then

$B(\phi, v) = B(v, \phi)$ must hold —— (7)

$$B(\phi, v) = \int_{-1}^{1} \frac{d\phi}{dx} v \, dx + \frac{1}{Pe} \int_{-1}^{1} \frac{d\phi}{dx} \frac{dv}{dx} dx \quad \text{—— (8)}$$

$$B(v, \phi) = \int_{-1}^{1} \frac{dv}{dx} \phi \, dx + \frac{1}{Pe} \int_{-1}^{1} \frac{dv}{dx} \frac{d\phi}{dx} dx$$

$$= -\int_{-1}^{1} \phi \frac{dv}{dx} dx + \frac{1}{Pe} \int_{-1}^{1} \frac{d\phi}{dx} \frac{dv}{dx} dx \quad \text{— (9)}$$

From (8) and (9), we conclude that

$$B(\phi, v) \neq B(v, \phi) \quad \text{—— (10)}$$

Hence $B(\phi, v)$ is not symmetric.

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Prob 2.8

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right)+cu=q(x) \quad \forall x\in\Omega=(0,1)\subset R^1 \quad -(1)$$

BCs: $u(0)=u_0$; $\left(a(x)\frac{du}{dx}+\beta(u-u_\infty)\right)\Big|_{x=1}=Q_1$ —— (2)

(a) Differential operator

$$A=-\frac{d}{dx}\left(a(x)\frac{d}{dx}\right)+c \quad \text{——} \quad (3)$$

$$\therefore \quad Au-f=0 \quad \forall x\in\Omega=(0,1)\subset R^1 \quad \text{——} \quad (4)$$

where $f=q(x)$

(b) linearity of A :

Let $u_1, u_2 \in D_A$

and $\alpha_1, \alpha_2 \in R$

Then

$$A(\alpha_1u_1+\alpha_2u_2)=\alpha_1Au_1+\alpha_2Au_2 \text{ must hold} \quad \text{——} \quad (5)$$

If the operator A is linear

Consider $A(\alpha_1u_1+\alpha_2u_2)=\left(-\frac{d}{dx}\left(a(x)\frac{d}{dx}\right)+c\right)(\alpha_1u_1+\alpha_2u_2)$

$$=\left(-\frac{d}{dx}\left(a(x)\frac{d}{dx}\right)+c\right)(\alpha_1u_1)+$$

$$\left(-\frac{d}{dx}\left(a(x)\frac{d}{dx}\right)+c\right)(\alpha_2u_2)$$

$$=\alpha_1\left(-\frac{d}{dx}\left(a(x)\frac{du_1}{dx}\right)+cu_1\right)+$$

$$\alpha_2\left(-\frac{d}{dx}\left(a(x)\frac{du_2}{dx}+cu_2\right)\right.$$

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$\therefore A(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 A u_1 + \alpha_2 A u_2$ —— (6)

Based on (4), (5) holds, hence the operator A is linear.

(c) Symmetry of A.

$\forall\ u, v \in D_A$ with $v = \delta u$

We must show that

$$(Au, v) = (u, Av) \quad \text{—— (6)}$$

if A is symmetric

Consider (Au, v)

$$(Au, v) = \int_0^1 \left(-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) + cu\right) v\, dx$$

$$= \int_0^1 -\frac{d}{dx}\left(a\frac{du}{dx}\right) v\, dx + \int_0^1 cuv\, dx \quad \text{—— (7)}$$

Perform integration by parts twice in the first term to transfer all differentiation from u to v.

IBP once give:

$$(Au, v) = \int_0^1 \left(\frac{dv}{dx} a(x)\right) \frac{du}{dx}\, dx - \left(\left(a(x)\frac{du}{dx}\right) v\right)\Big|_0^1 + \int_0^1 cuv\, dx \quad \text{—— (8)}$$

IBP once more (first term) and regroup terms.

$$(Au, v) = \int_0^1 \left(-\frac{d}{dx}\left(a(x)\frac{dv}{dx}\right) + cv\right) u\, dx + \left(\left(a(x)\frac{dv}{dx}\right) u\right)\Big|_0^1 -$$

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$$\left(\left(a(x) \frac{du}{dx} \right) v \right) \Big|_0^1 \qquad \text{——— (9)}$$

or

$$(Au, v) = (u, A^* v) + \langle Au, v \rangle_M. \qquad \text{——— (10)}$$

where $A^* = -\frac{d}{dx}\left(a(x) \frac{d}{dx} \right) + c$ ——— (11)

and $\langle Au, v \rangle_M = \left(a(x) \frac{dv}{dx} u \right) \Big|_0^1 - \left(a(x) \frac{du}{dx} v \right) \Big|_0^1$ ——— (12)

Simplification of Concomitant:

Consider Concomitant $\langle Au, v \rangle$ in (12). Expanding terms

$$\langle Au, v \rangle_M = a(1) u(1) \frac{dv}{dx}\Big|_1 - a(0) u(0) \frac{dv}{dx}\Big|_0$$

$$- a(1) v(1) \frac{du}{dx}\Big|_1 + a(0) v(0) \frac{du}{dx}\Big|_0 \qquad \text{——— (13)}$$

Recall BC's in (2):

$$u(0) = 0 \quad \Rightarrow \quad v(0) = 0 \qquad \text{——— (14)}$$

$$\left(a(x) \frac{du}{dx} \right)\Big|_1 = Q_1 - \beta(1) \left(u(1) - u_\infty(1) \right) \qquad \text{— (15)}$$

$$\therefore \; a(1) \frac{dv}{dx}\Big|_1 = -\beta(1) v(1) \qquad \text{——— (16)}$$

using (14), (15) and (16) in (13)

$$\langle Au, v \rangle_M = u(1) \left(-\beta(1) v(1) \right) - a(0) u_0 \frac{du}{dx}\Big|_0$$

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$$- v(1)\left(Q_1 - \beta(1)\left(u(1) - u_\infty(1)\right)\right) + a(0)\, v(0) \frac{du}{dx}\Big|_0$$

($v(0) \to 0$)

$$\therefore \langle Au, v\rangle_H = - \beta(1) u(1) v(1) - \left(a(0) \frac{du}{dx}\Big|_0\right) u_0$$

$$- v(1) Q_1 + \beta(1) u(1) v(1) - \beta(1) v(1) u_\infty(1)$$

$$\therefore \langle Au, v\rangle_H = - \left(a(0) \frac{du}{dx}\Big|_0\right) u_0 - v(1) Q_1 - \left(\beta(1) v(1)\right) u_\infty(1) \quad (17)$$

Hence, finally we have

$$(Au, v) = (u, A^* v) - \underbrace{\left(a(0) \frac{du}{dx}\Big|_0\right) u_0 - v(1) Q_1 - \beta(1) v(1) u_\infty(1)}_{\langle Au, v\rangle_H} \quad (18)$$

where

$$A^* = -\frac{d}{dx}\left(a(x) \frac{d}{dx}\right) + c = A \quad \text{——} \quad (19)$$

Concomitant $\langle Au, v\rangle$ in (17) is not zero if u_0, Q_1 and $u_\infty(1)$ are not zero i.e if the BCs are not homogeneous.

- Thus if u_0, Q_1 and $u_\infty(1)$ are not zero, the differential operator A is not symmetric
- A is symmetric if $u_0 = 0$, $Q_1 = 0$ and $u_\infty(1) = 0$ i.e if the BCs are homogeneous.

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Prob 2.9

$$-\frac{d}{dx}\left(b(x)\frac{d\phi}{dx}\right) = q(x) \quad \forall\, x \in \Omega = (0,L) \subset R^1 \qquad -(1)$$

BCs: $\phi(0) = 0$, $\left(b(x)\frac{d\phi}{dx} + k\phi\right)\Big|_{x=L} = P_L$ — (2)

(a) Differential operator

$$A = -\frac{d}{dx}\left(b(x)\frac{d}{dx}\right) \qquad — (3)$$

$$A\phi - f = 0\,; \quad f = q \quad \forall\, x \in (0,L) \subset R^1 \qquad — (4)$$

(b) Linearity of A

Let $\phi_1, \phi_2 \in D_A$

and $\alpha_1, \alpha_2 \in R$

Then $A(\alpha_1\phi_1 + \alpha_2\phi_2) = \alpha_1 A\phi_1 + \alpha_2 A\phi_2$ must hold — (5)

If the operator A is linear.

Consider $A(\alpha_1\phi_1 + \alpha_2\phi_2) = \left(-\frac{d}{dx}\left(b(x)\frac{d}{dx}\right)\right)(\alpha_1\phi_1 + \alpha_2\phi_2)$

$$= \left(-\frac{d}{dx}\left(b(x)\frac{d}{dx}\right)\right)(\alpha_1\phi_1) + \left(-\frac{d}{dx}\left(b(x)\frac{d}{dx}\right)\right)(\alpha_2\phi_2)$$

$$= \alpha_1\left(-\frac{d}{dx}\left(b(x)\frac{d\phi_1}{dx}\right)\right) + \alpha_2\left(-\frac{d}{dx}\left(b(x)\frac{d\phi_2}{dx}\right)\right)$$

$$= \alpha_1 A\phi_1 + \alpha_2 A\phi_2 \qquad — (6)$$

Based on (6), (5) holds, hence the operator A is linear.

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(c) Symmetry of A

$\forall \phi, v \in D_A$ with $v = \delta\phi$

we must show that

$$(A\phi, v) = (\phi, Av) \qquad \text{——— (7)}$$

if A is Symmetric

Consider $(A\phi, v)$

$$(A\phi, v) = \int_0^L \left(-\frac{d}{dx}\left(b(x)\frac{d\phi}{dx}\right)\right) v\, dx \qquad \text{——— (8)}$$

IBP once : transfr one order of differentiation from $b(x)\frac{d\phi}{dx}$ to v

$$(A\phi, v) = \int_0^L \left(\frac{dv}{dx} b(x)\right)\frac{d\phi}{dx} dx - \left(b(x)\frac{d\phi}{dx} v\right)\Big|_0^L \qquad \text{——— (9)}$$

IBP once more : transfr on order of differentiation from ϕ to $b(x)\frac{dv}{dx}$

$$(A\phi, v) = \int_0^L -\frac{d}{dx}\left(b(x)\frac{dv}{dx}\right)\phi\, dx + \left(\frac{dv}{dx} b(x)\phi\right)\Big|_0^L - \left(b(x)\frac{d\phi}{dx} v\right)\Big|_0^L \qquad \text{——— (10)}$$

$$\therefore (A\phi, v) = (\phi, A^* v) + \langle A\phi, v\rangle_\mu \qquad \text{——— (11)}$$

In which

$$A^* = -\frac{d}{dx}\left(b(x)\frac{d}{dx}\right) \qquad \text{——— (12)}$$

$$\langle A\phi, v\rangle_\mu = \left(\frac{dv}{dx} b(x)\phi\right)\Big|_0^L - \left(b(x)\frac{d\phi}{dx} v\right)\Big|_0^L \qquad \text{——— (13)}$$

Simplification of Concomitant

Expanding the boundary terms in (13)

$$\langle A\phi, v\rangle_\mu = \left(\frac{dv}{dx}\Big|_L b(L)\right)\phi(L) - \frac{dv}{dx}\Big|_0 b(0)\phi(0) - b(L)\frac{d\phi}{dx}\Big|_L v(L) + b(0)\frac{d\phi}{dx}\Big|_0 v(0) \qquad \text{——— (14)}$$

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Recall BCs in (2):

$$\left.\begin{aligned} &\phi(0) = 0 \quad \Rightarrow \quad v(0) = 0 \\ &\left(b(x)\frac{d\phi}{dx}\right)\Big|_L = P_L - (k\phi)\Big|_L \quad \Rightarrow \quad b(L)\frac{dv}{dx}\Big|_L = -kv(L) \end{aligned}\right\} \text{—(15)}$$

Consider Concomitant (14) and substitute from

$$\langle A\phi, v\rangle_\mu = \left(-kv(L)\right)\phi(L) - \frac{dv}{dx}\Big|_0 \underbrace{b(0)\phi(0)}_{\to 0} - \left(P_L - k\phi(L)\right)v(L) + \underbrace{b(0)\frac{d\phi}{dx}\Big|_0 v(0)}_{\to 0}$$

$$- kv(L)\phi(L) + kv(L)\phi(L) - P_L v(L) \quad \text{—(16)}$$

$$\therefore \langle A\phi, v\rangle_\mu = -P_L v(L) \quad \text{——— (17)}$$

i.e. Concomitant $\langle A\phi, v\rangle_\mu \neq 0$ if $P_L \neq 0$ i.e $\langle A\phi, v\rangle_\mu$ is only zero if the BCs are homogeneous.

Adjoint of A

$$A^* = -\frac{d}{dx}\left(b(x)\frac{d}{dx}\right) = A. \quad \text{——— (18)}$$

based on (17) and (18) The differential operator A is symmetric only if $P_L = 0$ i.e The BCs are homogeneous.

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Prob 2.10

$$\frac{d^2}{dx^2}\left(b(x)\frac{d^2\psi}{dx^2}\right)+\frac{d}{dx}\left(c(x)\frac{d\psi}{dx}\right)+a(x)\psi=f(x) \quad \text{——(1)}$$

$$\forall\, x\in\Omega=(0,L)\subset R^1$$

BCs:

$$\left.\begin{aligned}\psi(0)&=\psi_0\\ \psi(L)&=\psi_L\\ \left(b(x)\frac{d^2\psi}{dx^2}\right)\Big|_{x=0}&=M_0\\ \left(b(x)\frac{d^2\psi}{dx^2}\right)\Big|_{x=L}&=M_L\end{aligned}\right\} \quad \text{——(2)}$$

(a) differential operator A

$$A=\frac{d^2}{dx^2}\left(b(x)\frac{d^2}{dx^2}\right)+\frac{d}{dx}\left(c(x)\frac{d}{dx}\right)+a(x) \quad \text{——(3)}$$

$$\therefore\ A\psi-f=0\ \forall\, x\in(0,L)\subset R^1 \quad \text{——(4)}$$

(b) Linearity of A

Let $\psi_1,\psi_2\in D_A$
and $\alpha_1,\alpha_2\in R$

Then $A(\alpha_1\psi_1+\alpha_2\psi_2)=\alpha_1 A\psi_1+\alpha_2 A\psi_2$ must hold —(5)

If the operator A is linear.

Consider $A(\alpha_1\psi_1+\alpha_2\psi_2)=\left(\frac{d^2}{dx^2}\left(b(x)\frac{d^2}{dx^2}\right)+\frac{d}{dx}\left(c(x)\frac{d}{dx}\right)+a(x)\right)$

$$(\alpha_1\psi_1+\alpha_2\psi_2) \quad \text{——(5)}$$

or

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$$A(\alpha_1\psi_1+\alpha_2\psi_2) = \left(\frac{d^2}{dx^2}\left(b(x)\frac{d^2}{dx^2}\right)+\frac{d}{dx}\left(c(x)\frac{d}{dx}\right)+a(x)\right)(\alpha_1\psi_1)+$$

$$\left(\frac{d^2}{dx^2}\left(b(x)\frac{d^2}{dx^2}\right)+\frac{d}{dx}\left(c(x)\frac{d}{dx}\right)+a(x)\right)(\alpha_2\psi_2)$$

$$= \alpha_1\left(\frac{d^2}{dx^2}\left(b(x)\frac{d^2\psi_1}{dx^2}\right)+\frac{d}{dx}\left(c(x)\frac{d\psi_1}{dx}\right)+a(x)\psi_1\right)+$$

$$\alpha_2\left(\frac{d^2}{dx^2}\left(b(x)\frac{d^2\psi_2}{dx^2}\right)+\frac{d}{dx}\left(c(x)\frac{d\psi_2}{dx}\right)+a(x)\psi_2\right)$$

$$= \alpha_1 A\psi_1 + \alpha_2 A\psi_2 \qquad \text{——— (6)}$$

Based on (6), (5) holds, hence the operator A is linear.

(c) Symmetry of operator A

$\forall\ \psi, v \in D_A$ with $v = \delta\psi$

we must show that

$$(A\psi, v) = (\psi, Av) \qquad \text{——— (7)}$$

if A is symmetric

Consider $(A\psi, v)$

$$(A\psi, v) = \int_0^L \left(\frac{d^2}{dx^2}\left(b(x)\frac{d^2\psi}{dx^2}\right)+\frac{d}{dx}\left(c(x)\frac{d\psi}{dx}\right)+a(x)\psi\right)v\,dx$$

$$= \int_0^L \frac{d^2}{dx^2}\left(b(x)\frac{d^2\psi}{dx^2}\right)v\,dx + \int_0^L \frac{d}{dx}\left(c(x)\frac{d\psi}{dx}\right)v\,dx +$$

$$\int_0^L a(x)\psi v\,dx \qquad \text{——— (8)}$$

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Consider the first term on the right side of (8).

$$\int_0^L \frac{d^2}{dx^2}\left(b(x)\frac{d^2\psi}{dx^2}\right) v\,dx = \underset{\text{(IBP twice)}}{\int_0^L \left(\frac{d^2 v}{dx^2} b(x)\right)\frac{d^2\psi}{dx^2}\,dx} + \left(\frac{d}{dx}\left(b(x)\frac{d^2\psi}{dx^2}\right)v\right)\Big|_0^L$$

$$- \left(b(x)\frac{d^2\psi}{dx^2}\frac{dv}{dx}\right)\Big|_0^L \quad \text{—— (9)}$$

Transfer two orders of differentiation from ψ to $\frac{d^2 v}{dx^2} b(x)$ in the first term on right side of (9).

$$\therefore \int_0^L \frac{d^2}{dx^2}\left(b(x)\frac{d^2\psi}{dx^2}\right)v\,dx = \int_0^L \frac{d^2}{dx^2}\left(b(x)\frac{d^2 v}{dx^2}\right)\psi\,dx + \left(\frac{d^2 v}{dx^2} b(x)\frac{d\psi}{dx}\right)\Big|_0^L$$

$$- \left(\frac{d}{dx}\left(b(x)\frac{d^2 v}{dx^2}\right)\psi\right)\Big|_0^L + \left(\frac{d}{dx}\left(b(x)\frac{d^2\psi}{dx^2}\right)v\right)\Big|_0^L$$

$$- \left(b(x)\frac{d^2\psi}{dx^2}\frac{dv}{dx}\right)\Big|_0^L \quad \text{—— (10)}$$

Consider second term on right side of (8)

$$\int_0^L \frac{d}{dx}\left(c(x)\frac{d\psi}{dx}\right)v\,dx \underset{\text{(IBP twice)}}{=} \int_0^L \frac{d}{dx}\left(c(x)\frac{dv}{dx}\right)\psi\,dx$$

$$+ \left(c(x)\frac{d\psi}{dx}v\right)\Big|_0^L - \left(c(x)\frac{dv}{dx}\psi\right)\Big|_0^L \quad \text{(11)}$$

using (10) and (11) in (8)

$$(A\psi, v) = \int_0^L \left(\frac{d^2}{dx^2}\left(b(x)\frac{d^2 v}{dx^2}\right) + \frac{d}{dx}\left(c(x)\frac{dv}{dx}\right) + dv\right)\psi\,dx$$

$$+ \langle A\psi, v\rangle_\Gamma \quad \text{—— (12)}$$

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$$(A\psi, v) = (\psi, A^* v) + \langle A\psi, v \rangle_M \qquad \text{—— (13)}$$

in which

$$A^* = \frac{d^2}{dx^2}\left(b(x)\frac{d^2}{dx^2}\right) + \frac{d}{dx}\left(c(x)\frac{d}{dx}\right) + d(x) = A$$

Simplification of Concomitant $\langle A\psi, v \rangle_M$

$$\langle A\psi, v \rangle_M = \left(\frac{d^2 v}{dx^2} b(x) \frac{d\psi}{dx}\right)\Big|_0^L - \left(\frac{d}{dx}\left(b(x)\frac{d^2 v}{dx^2}\right)\psi\right)\Big|_0^L$$
$$+ \left(\frac{d}{dx}\left(b(x)\frac{d^2\psi}{dx^2}\right) v\right)\Big|_0^L - \left(\left(b(x)\frac{d^2\psi}{dx^2}\right)\frac{dv}{dx}\right)\Big|_0^L$$
$$+ \left(c(x)\frac{d\psi}{dx} v\right)\Big|_0^L - \left(c(x)\frac{dv}{dx}\psi\right)\Big|_0^L \qquad \text{—— (14)}$$

Recall BCs in (2)..

$$\left.\begin{aligned}
&\psi(0) = \psi_0 \quad \Rightarrow \quad v(0) = 0 \\
&\psi(L) = \psi_L \quad \Rightarrow \quad v(L) = 0 \\
&\left(b(x)\frac{d^2\psi}{dx^2}\right)\Big|_{x=0} = M_0 \quad \Rightarrow \quad b(0)\frac{d^2 v}{dx^2}\Big|_0 = 0 \\
&\left(b(x)\frac{d^2\psi}{dx^2}\right)\Big|_{x=L} = M_L \quad \Rightarrow \quad b(L)\frac{d^2 v}{dx^2}\Big|_L = 0
\end{aligned}\right\} \qquad \text{—— (15)}$$

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We expand boundary terms in (14) and then use (15)

$$\langle A\psi, v\rangle_H = \left(\frac{d^2v}{dx^2} b(x) \frac{d\psi}{dx}\right)\Big|_L - \left(\frac{d^2v}{dx^2} b(x) \frac{d\psi}{dx}\right)\Big|_0$$

$$- \left(\frac{d}{dx}\left(b(x) \frac{d^2v}{dx^2}\right)\psi\right)\Big|_L + \left(\frac{d}{dx}\left(b(x) \frac{d^2v}{dx^2}\right)\psi\right)\Big|_0$$

$$+ \left(\frac{d}{dx}\left(b(x) \frac{d^2\psi}{dx^2}\right)v\right)\Big|_L - \left(\frac{d}{dx}\left(b(x) \frac{d^2\psi}{dx^2}\right)v\right)\Big|_0$$

$$- \left(b(x) \frac{d^2\psi}{dx^2} \frac{dv}{dx}\right)\Big|_L + \left(b(x) \frac{d^2\psi}{dx^2} \frac{dv}{dx}\right)\Big|_0$$

$$\left(c(x) \frac{d\psi}{dx} v\right)\Big|_L - \left(c(x) \frac{d\psi}{dx} v\right)\Big|_0$$

$$- \left(c(x) \frac{dv}{dx} \psi\right)\Big|_L + \left(c(x) \frac{dv}{dx} \psi\right)\Big|_0 \quad \text{—— (16)}$$

$\therefore$

$$\langle A\psi, v\rangle_H = - \left(\frac{d}{dx}\left(b(x) \frac{d^2v}{dx^2}\right)\right)\Big|_L \psi_L + \left(\frac{d}{dx}\left(b(x) \frac{d^2v}{dx^2}\right)\right)\Big|_0 \psi_0$$

$$- \left(\frac{dv}{dx}\right)\Big|_L M_L + \left(\frac{dv}{dx}\right)\Big|_0 M_0$$

$$- \left(c(x) \frac{dv}{dx}\right)\Big|_L \psi_L + \left(c(x) \frac{dv}{dx}\right)\Big|_0 \psi_0 \quad \text{—— (17)}$$

This is the final simplified form of the Concomitant $\langle A\psi, v\rangle_H$.

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1. $\langle A\psi, v \rangle_\pi$ goes to zero if $\psi_0 = \psi_L = M_0 = M_L = 0$ i.e if the boundary conditions are homogeneous otherwise it is nonzero.

2. $A^* = A$ i.e the adjoint of the operator is same as the operator

3. Based on 1 and 2 The differential operator A is not symmetric with the BCs in (2)

4. If all BCs are homogeneous, then A is symmetric

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Prob. 2.11

$$-\frac{d}{dx}\left(\phi \frac{d\phi}{dx}\right) + +(x) = 0 \quad \forall \; x \in \Omega = (0,1) \subset R^1 \qquad (1)$$

BCs: $\left.\frac{d\phi}{dx}\right|_{x=0} = q_0$ and $\phi(1) = \sqrt{2}$ —— (2)

(a) Differential operator A

$$A = -\frac{d}{dx}\left(\phi \frac{d}{dx}\right) \qquad \text{——— (3)}$$

$$\therefore \quad A\phi - f = 0 \quad \forall \; x \in (0,1) \subset R^1 \qquad \text{— (4)}$$

(b) Linearity of A

Let $\phi_1, \phi_2 \in D_A$

and $\alpha_1, \alpha_2 \in R$

Then $A(\alpha_1\phi_1 + \alpha_2\phi_2) = \alpha_1 A\phi_1 + \alpha_2 A\phi_2$ must hold —— (5)

if the operator A is linear

Consider $A(\alpha_1\phi_1 + \alpha_2\phi_2) = -\frac{d}{dx}\left((\alpha_1\phi_1 + \alpha_2\phi_2)\frac{d}{dx}(\alpha_1\phi_1 + \alpha_2\phi_2)\right)$ (6)

and

$$\alpha_1 A\phi_1 = -\alpha_1 \frac{d}{dx}\left(\phi_1 \frac{d\phi_1}{dx}\right) \qquad \text{—— (7)}$$

$$\alpha_2 A\phi_2 = -\alpha_2 \frac{d}{dx}\left(\phi_2 \frac{d\phi_2}{dx}\right) \qquad \text{—— (8)}$$

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Clearly Sum of (7) and (8) is not equal to (6), hence for this operator A (5) does not hold.

Therefore the differential operator A in (3) is not linear.

- Since the differential operator is not linear it cannot be symmetric.
- In this case A^* is not meaningful.

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