

Differential Equations for Engineers The Essentials
1st edition Kalbaugh Solutions Manual

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Differential Equations for Engineers: the Essentials – Example Course Plan

Date	Class #	# of slides	Topics	Reading Assigned	Problems Assigned	Problems Reviewed
	1	36	Administrative matters, course objectives, importance of DEs to engineers, review of math foundations, classes of DEs	Chap 1 (14 pages)		–
	2	31	First order linear ODEs: RC circuit; general solution to homogeneous eqns; in-class homogeneous problems; water pipe temperature example; general solution nonhomogeneous; In-class nonhomogeneous problems	Chap 2 (11 pages)		–
	3	35	1st order linear ODEs: System viewpt First order nonlinear separable ODEs: General solution approach; In-class problems; Sounding rocket phases 1 & 2	Chap 3 thru Sec 3.1 (7 pages)		From Class #1
	4	21	First order nonlinear separable ODEs: Sounding rocket phase 3; Short quiz #1	Chap 3, Sec 3.2, 3.3 (5 pages)		From Class #2
	5	32	Review of short quiz #1 First order ODEs: successive approximations with example; in-class problems; existence and uniqueness	Chap 4, thru Sec 4.4 (9 pages)		From Class #3
	6A/B	30/36	Qualitative analysis Stability revisited Computing project phase 1	Chap 4, Sec 4.5, 4.6 (3 pages)	Computing project phase 1	From Class #4
	7	24	2nd order LTI homogeneous ODEs LRC circuit, characteristic equation, real and repeated roots; in-class problems	Chap 5 thru Sec 5.1.2 (6 pages)		From Class #5
	8		Test #1			–
	9	28	Review of Test #1 2nd order LTI homogeneous ODEs, LRC circuit, complex roots, fundamental solutions, In-class example problems	Chap 5, Sec 5.1.3, 5.1.4, 5.2 (5 pages)		From Class #7
	10	41	2nd order LTI Nonhomogeneous ODEs; Cruise control Undetermined coefficients method LRC circuit with sine source	Chap 5, Sec 5.3, 5.4 (15 pages)		From Class #6
	11	42	Process & example for kernel method Undetermined coefficients example Higher order ODEs	Chap 6 (10 pages)		From Class #9

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			Satellite orbit decay			
	12		Review for Midterm			From Classes #10, 11
	13		Mid-term exam			—
	14A/B	42/44	Review of mid-term Laplace transforms: intro, homogeneous equations Computing project phase 2	Chap 7 thru Sec 7.5 (11 pages)	Computing project phase 2	
	15	46	Laplace transforms: Nonhomogeneous equations et al	Chap 7, Sec 7.6 (20 pages)		
	16	43	State space format: Numerical methods Review of matrix algebra Linear systems in state space format	Chap 8 thru Sec 8.4 (11 pages)		
	17	20	State space format: Heat transfer Short quiz #2	Chap 8, Sec 8.5.1.1 (8 pages)		From class #15
	18	35	Review of short quiz #2 State space format: Two-state electrical circuit; Aircraft dynamics	Chap 8, Sec 8.5.1.2, 8.5.1.3 (15 pages)		From classes #14,#16
	19	42	Three state electrical circuit Repeated eigenvalues; Coordinate systems: Vehicle suspension system	Chap 8, Sec 8.5.1.4 thru 8.5.1.7 (12 pages)		From classes #17,#18
	20		Test #2			—
	21	37	Review of Test #2 Coordinate systems; state transition matrix; nonhomogeneous equations, kernel method, 2-state electrical circuit example	Chap 8, Sec 8.5.1.5 thru 8.5.2.1 (15 pages)		From class #19
	22	35	Nonhomogeneous equations: Laplace transform method, trial and error method, PDEs: IV heat equation	Chap 8, Sec 8.5.2.2, 8.5.2.3 (6 pages)		
	23	24	PDEs: BV heat equation – Fourier series Short quiz #3	Chap 9 thru Sec 9.1 (11 pages)		From class #21
	24	35	Review of Short quiz #3 PDEs: wave equation; IV wave equation, BV problem: membrane – power series I	Chap 9, Sec 9.2 thru 9.2.3 (16 pages)		From class #22
	25	35	PDEs: higher order Bessell functions;	Chap 9,		From class

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			power series II	Sec 9.2.4 thru 9.2.6 (8 pages)		#23
	26	34	PDEs: BV potential equation – Legendre's eq'n; cantilever beam	Chap 9, Sec 9.3, 9.4 (13 pages)		From class #24, #25
	27		Test #3			
	28		Review of Test #3 Review for final exam			
	29		Review for final exam			
			Final Exam			

Differential Equations for Engineers: the Essentials

Supplement to Class 4 Notes

Contents of Supplement

Differences Between Linear and Nonlinear ODEs in Their Input / Output Response

Example: Nonlinear Circuit

Differences Between Linear and Nonlinear ODEs in Their Input / Output Response

Input / Output Characteristics for a Linear System

In a linear system of any order (time-varying or time-invariant):

If the output is $y_1(t)$ when the input is $u_1(t)$ and the output is $y_2(t)$
when the input is $u_2(t)$
then the output is $ay_1(t) + by_2(t)$ when the input is $au_1(t) + bu_2(t)$
for any constants a and b

In a linear time-invariant system of any order, after transients have died away, when the input is a sine-wave of a given frequency the output is a steady-state oscillation of only that frequency.

These characteristics are not generally true of nonlinear systems

Existence and Uniqueness of Solutions to Linear ODEs

Theorem: Given a linear nth order ODE:

$$\frac{d^n y}{dt^n} + a_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_{n-1}(t) \frac{dy}{dt} + a_n(t) y = g(t)$$

with initial conditions

$$\frac{d^{n-1} y}{dt^{n-1}}(0) = y_0^{(n-1)} \quad \frac{d^{n-2} y}{dt^{n-2}}(0) = y_0^{(n-2)} \quad \dots \quad \frac{dy}{dt}(0) = y_0' \quad y(0) = y_0$$

If the coefficients $a_i(t)$ and the input $g(t)$ are continuous for all t then there exists a unique solution to the ODE satisfying the initial conditions for all t .

(Stated without proof.)

This is not generally true of nonlinear ODEs.

Key Points from the Following Nonlinear Example

The method of successive approximations (solving a sequence of linear equations to approximate the solution of a nonlinear equation) is a powerful tool.

Nonlinearities in systems designed to be linear cause distortions in the frequency response, introducing “harmonics” (oscillations that are multiples of the input frequency).

Nonlinear Example: LR Circuit

Kirchhoff's Law:

Sum of voltage drops around a closed circuit = 0

Voltage drop over an inductor:

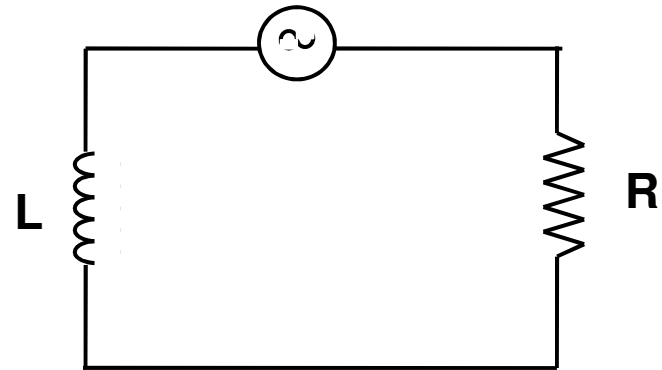
$$V = L \frac{dI}{dt}$$

Voltage drop over a resistor:

$$V = IR$$

Voltage drop over source:

$$V = -V_0 \sin(\omega t)$$



Resulting equation :

$$L \frac{dI}{dt} + IR = V_0 \sin(\omega t)$$

Nonlinear Example: LR Circuit (2)

The solution to

$$L \frac{dI}{dt} + IR = V_0 \sin(\omega t)$$

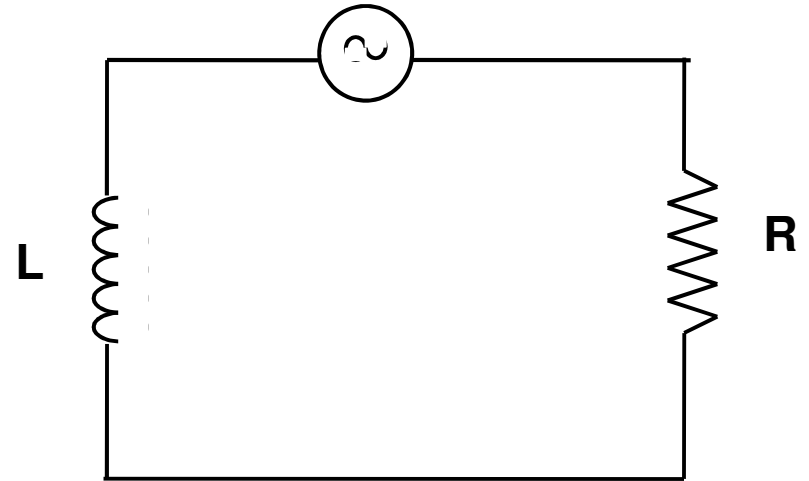
$$I(0) = 0$$

is

$$I(t) = \frac{\lambda}{\sqrt{\lambda^2 + \omega^2}} (V_0 / R) \sin(\omega t - \theta_1) + \frac{\lambda \omega}{\lambda^2 + \omega^2} (V_0 / R) e^{-\lambda t}$$

where

$$\theta_1 = \arctan(\omega / \lambda) \quad \lambda = R / L$$



Nonlinear Example: Recalling the LR Circuit (3)

Input / output response:

The “input” to the “system” is the voltage source. The “output” is the current (or voltage) over the resistor.

Ignoring the transient, the system passes the sine wave frequency perfectly - it introduces no other frequencies.

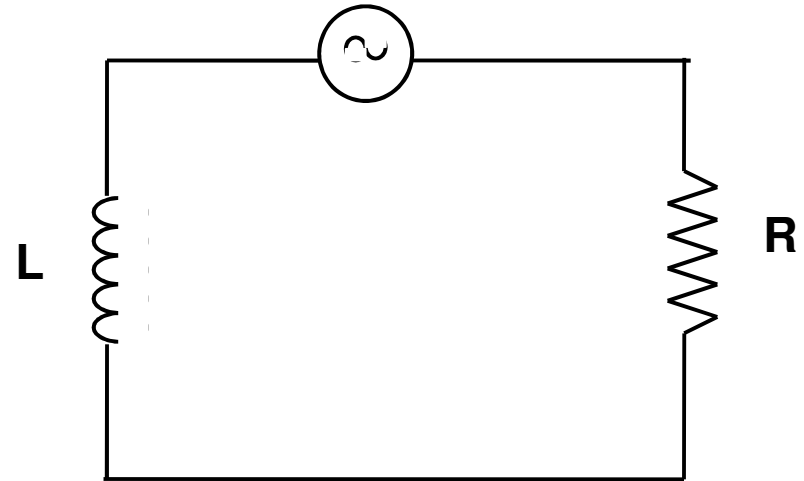
We only discuss frequency response in the context of time-invariant systems.

What if the Resistor is Slightly Nonlinear?

Instead of

$$L \frac{dI}{dt} + IR = V_0 \sin(\omega t)$$

suppose we have

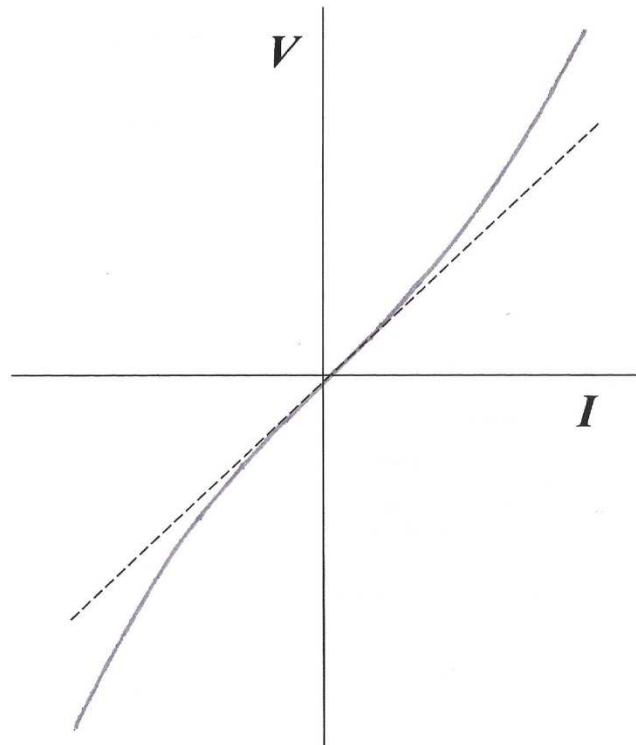


$$L \frac{dI}{dt} + R(I + \epsilon I^3) = V_0 \sin \omega t$$

$$\frac{dI}{dt} + \lambda(I + \epsilon I^3) = \lambda \left(\frac{V_0}{R} \right) \sin(\omega t) = \lambda I_0 \sin(\omega t)$$

Equation 1

What if the Resistor is Slightly Nonlinear? (2)



Nonlinear resistor characteristic

Approach to Solution of Slightly Nonlinear ODE

Since ε is small, consider the successive approximations

$$\frac{dI_1}{dt} + \lambda I_1 = \lambda I_0 \sin(\omega t) \quad \text{Equation 2}$$

$$I_1(0) = 0$$

$$\frac{dI_2}{dt} + \lambda I_2 = \lambda I_0 \sin(\omega t) - \varepsilon \lambda I_1^3(t) \quad \text{Equation 3}$$

$$I_2(0) = 0$$

$$\frac{dI_3}{dt} + \lambda I_3 = \lambda I_0 \sin(\omega t) - \varepsilon \lambda I_2^3(t) \quad \text{Equation 4}$$

$$I_3(0) = 0$$

Approach to Solution of Slightly Nonlinear ODE (2)

We are solving the nonlinear Equation 1 approximately by the sequential solution of a series of linear Equations 2, 3 and 4.

The general solution of

$$\frac{dI}{dt} + \lambda I = g(t) \quad \text{is} \quad I(t) = \int_0^t e^{-\lambda(t-\tau)} g(\tau) d\tau \quad \text{Equation 5}$$
$$I(0) = 0$$

Using Equation 5, we have already found that the solution to Equation (2) is

$$I_1(t) = \frac{\lambda}{\sqrt{\lambda^2 + \omega^2}} I_0 \sin(\omega t - \theta_1) + \frac{\lambda \omega}{\lambda^2 + \omega^2} I_0 e^{-\lambda t}$$

In what follows we will ignore the transient. We are examining the steady state frequency response.

Approach to Solution of Slightly Nonlinear ODE (3)

Repeating Equation 3:

$$\frac{dI_2}{dt} + \lambda I_2 = \lambda I_0 \sin \omega t - \lambda \varepsilon I_1^3$$

This has solution

$$I_2(t) = \int_0^t e^{-\lambda(t-\tau)} (\lambda I_0 \sin(\omega\tau) - \lambda \varepsilon I_1^3) d\tau$$

Now

$$\int_0^t e^{-\lambda(t-\tau)} \lambda I_0 \sin(\omega\tau) d\tau = I_1(t)$$

so

$$I_2(t) = I_1(t) - \lambda \varepsilon \int_0^t e^{-\lambda(t-\tau)} I_1^3(\tau) d\tau$$

Approach to Solution of Slightly Nonlinear ODE (4)

Continuing in this way, we find

$$I_{n+1}(t) = I_1(t) - \lambda \varepsilon \int_0^t e^{-\lambda(t-\tau)} I_n^3(\tau) d\tau$$

which one can show can be written as

$$I_n(t) = I_1(t) - G_1 I_0 \eta_n(t)$$

where

$$G_1 = \frac{\lambda}{\sqrt{\lambda^2 + \omega^2}} \qquad I_0 = V_0 / R$$

$$\eta_n(t) = \sum_{m=1}^{N_n} k_{mn} \sin(m\omega t - \phi_{mn})$$


Note the higher frequencies – the harmonics

Approach to Solution of Slightly Nonlinear ODE (4)

Challenge problem:

Given the steady state solution

$$I_1(t) = G_1 I_0 \sin(\omega t - \theta_1)$$

find the steady state component of $I_2(t)$

Differential Equations for Engineers: the Essentials

Class 2 notes

Agenda: Class 2

First order linear differential equations:

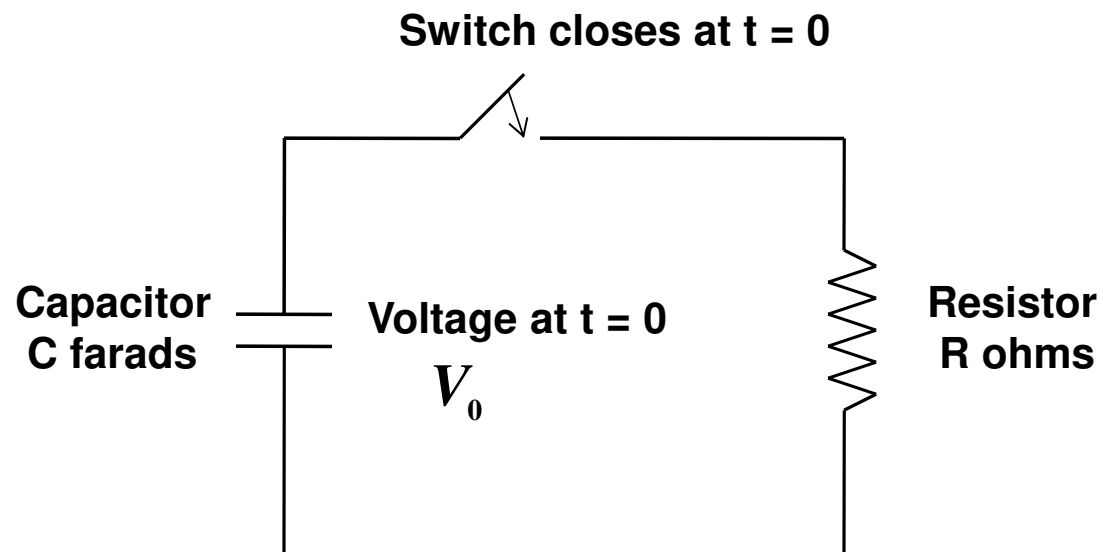
- (1) Engineering example: RC circuit**
- (2) General solution of homogeneous equation**
- (3) In-class homogeneous problems**
- (4) Example: Exposed water pipe in cyclical air temperature**
- (5) General solution of nonhomogeneous equation**
- (6) In-class nonhomogeneous problems**

Homework Assignment 2

First Order Linear Differential Equations

Example: The RC Electrical Circuit

Example: RC Electrical Circuit



Example: RC Circuit (2)

Kirchhoff's Law:

Sum of voltage drops around a closed circuit = 0

Voltage drop over a capacitor:

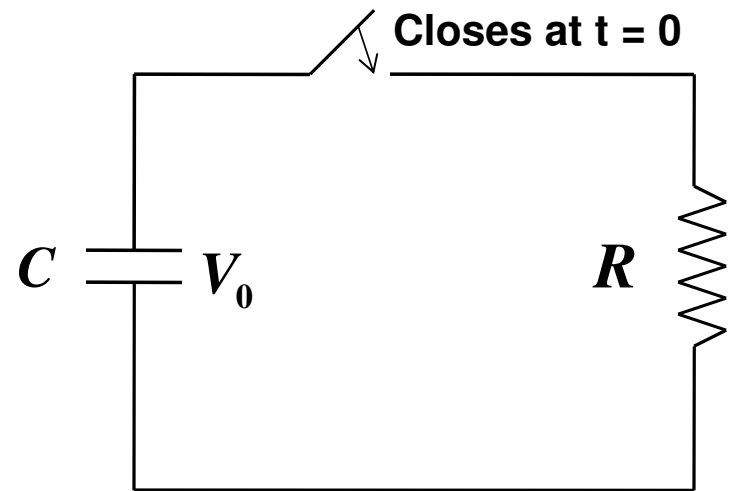
$$V = q / C \quad q = \text{charge on capacitor}$$

Voltage drop over a resistor:

$$V = IR \quad I = \text{current through resistor}$$

Conservation of electrical charge:

$$\frac{dq}{dt} = I$$



Example: RC Circuit (3)

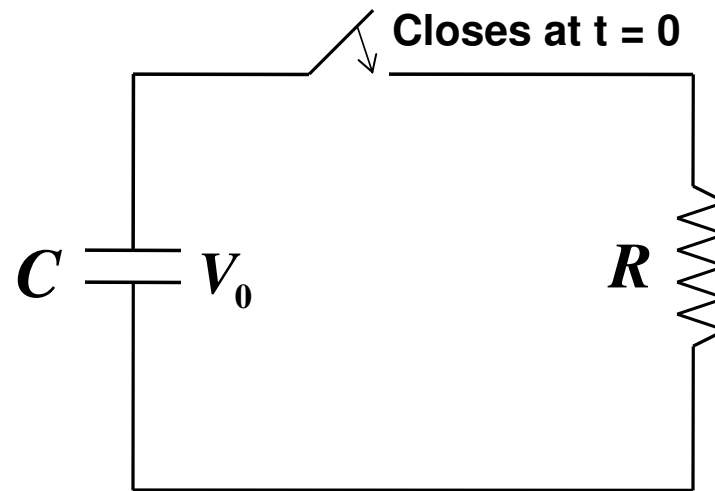
Resulting differential equation

$$IR + V = 0$$

$$\frac{dq}{dt}R + V = 0$$

$$RC \frac{dV}{dt} + V = 0$$

$$\frac{dV}{dt} + \frac{1}{RC}V = 0$$



Initial condition:

$$V(0) = V_0$$

Example: RC Circuit (4)

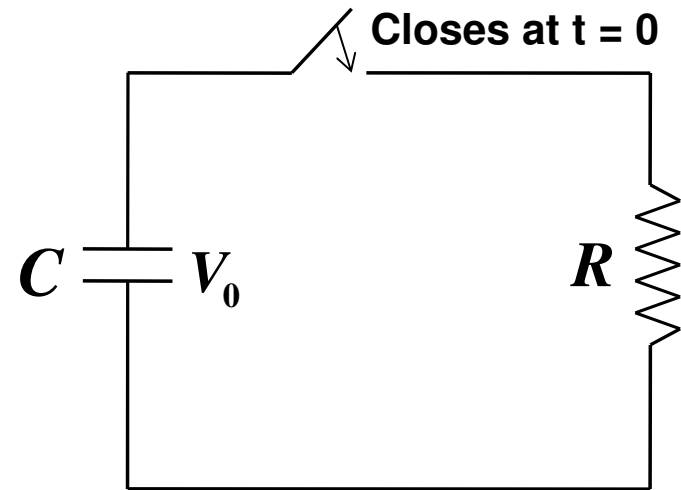
This is a linear first order ODE. To solve it, we separate variables: i.e., we put all terms involving V on the left side and all terms involving t on the right: specifically, we divide by V , move $1/RC$ to the right side and multiply by dt :

$$\frac{dV}{dt} + \frac{1}{RC}V = 0$$

$$\frac{1}{V} \frac{dV}{dt} + \frac{1}{RC} = 0$$

$$\frac{1}{V} \frac{dV}{dt} = -\frac{1}{RC}$$

$$\frac{dV}{V} = -\frac{1}{RC}dt$$



The text justifies this short-cut procedure

Example: RC Circuit (5)

Next, we integrate both sides:

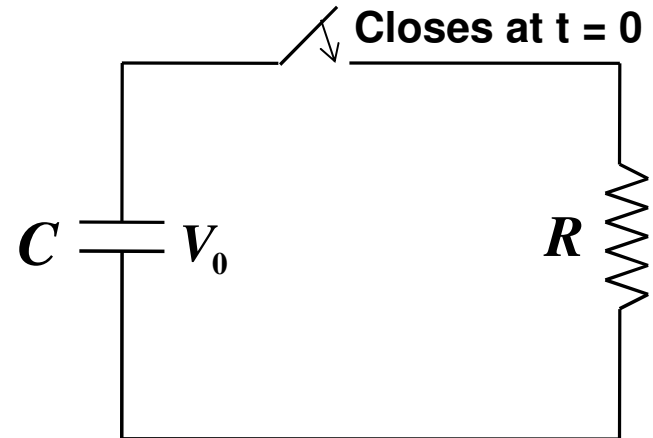
$$\int_{V(0)}^{V(t)} \frac{dV}{V} = -\int_0^t \frac{dt}{RC}$$

$$\ln\left(\frac{V(t)}{V(0)}\right) = -t / RC$$

Taking the exponential of both sides:

$$\exp\left(\ln\left(\frac{V(t)}{V(0)}\right)\right) = \exp(-t / RC)$$

$$\frac{V(t)}{V(0)} = e^{-t/RC}$$



Example: RC Circuit (6)

Hence:

$$V(t) = V(0)e^{-t/RC}$$

We require that the voltage over the capacitor at time 0 be given by

$$V(0) = V_0$$

and so

$$V(t) = V_0e^{-t/RC}$$

Example: RC Circuit (7)

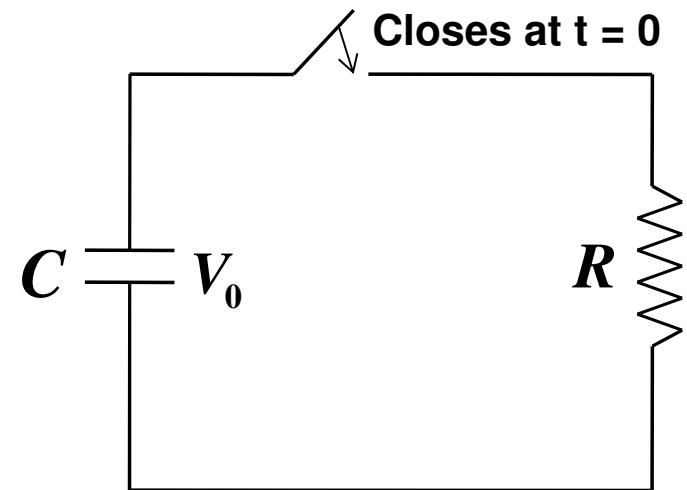
In summary, the solution to

$$\frac{dV}{dt} + \frac{1}{RC}V = 0$$

$$V(0) = V_0$$

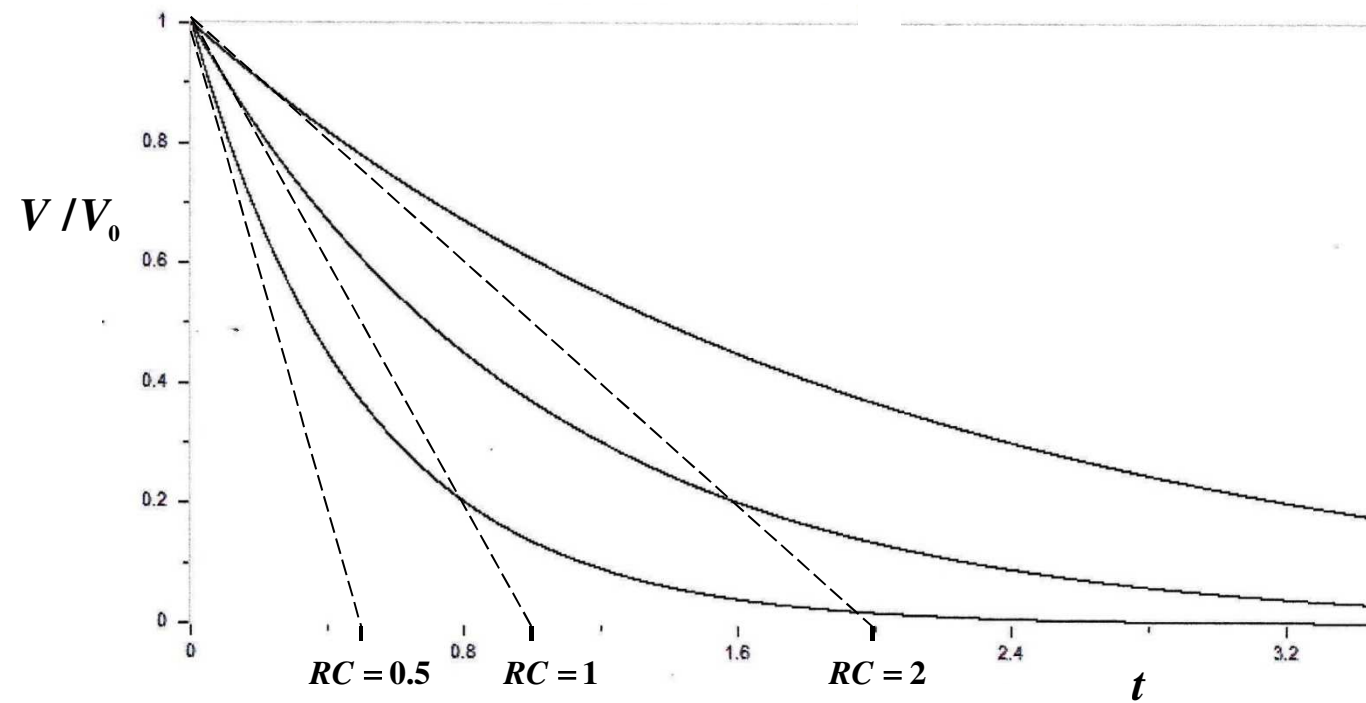
is

$$V(t) = V_0 e^{-t/RC}$$



The product RC has the dimension of time and is called the time constant for the circuit

Example: RC Circuit (8)



First Order Linear Differential Equations

General Solution of Homogeneous Equation

General Solution of Homogeneous Equation

The general form of a first order linear time-varying ordinary differential equation (ODE) is

$$\frac{dy}{dt} + p(t)y = 0 \qquad y(t_0) = y_0 \qquad \text{Equation 1}$$

How do we solve it?

General Solution of Homogeneous Equation (2)

We separate variables

$$\frac{dy}{y} = -p(t)dt$$

Integrate both sides

$$\int_{y_0}^{y(t)} \frac{dy}{y} = -\int_{t_0}^t p(\tau) d\tau$$

$$\ln\left(\frac{y(t)}{y_0}\right) = -\int_{t_0}^t p(\tau) d\tau$$

Take the exponential of both sides

$$\exp\left(\ln\left(\frac{y(t)}{y_0}\right)\right) = \exp\left(-\int_{t_0}^t p(\tau) d\tau\right)$$

$$\frac{y(t)}{y_0} = \exp\left(-\int_{t_0}^t p(\tau) d\tau\right)$$

General Solution of Homogeneous Equation (3)

Summary:

The solution of

$$\frac{dy}{dt} + p(t)y = 0 \qquad y(t_0) = y_0$$

is

$$y(t) = y_0 \exp\left(-\int_{t_0}^t p(\tau) d\tau\right)$$

Success depends entirely on being able to do the integral

Homogeneous First Order Linear ODEs: In-class problems

$$\frac{dy}{dt} + ky = 0$$

$$y(0) = a$$

$$\frac{dy}{dt} + ty = 0$$

$$y(0) = b$$

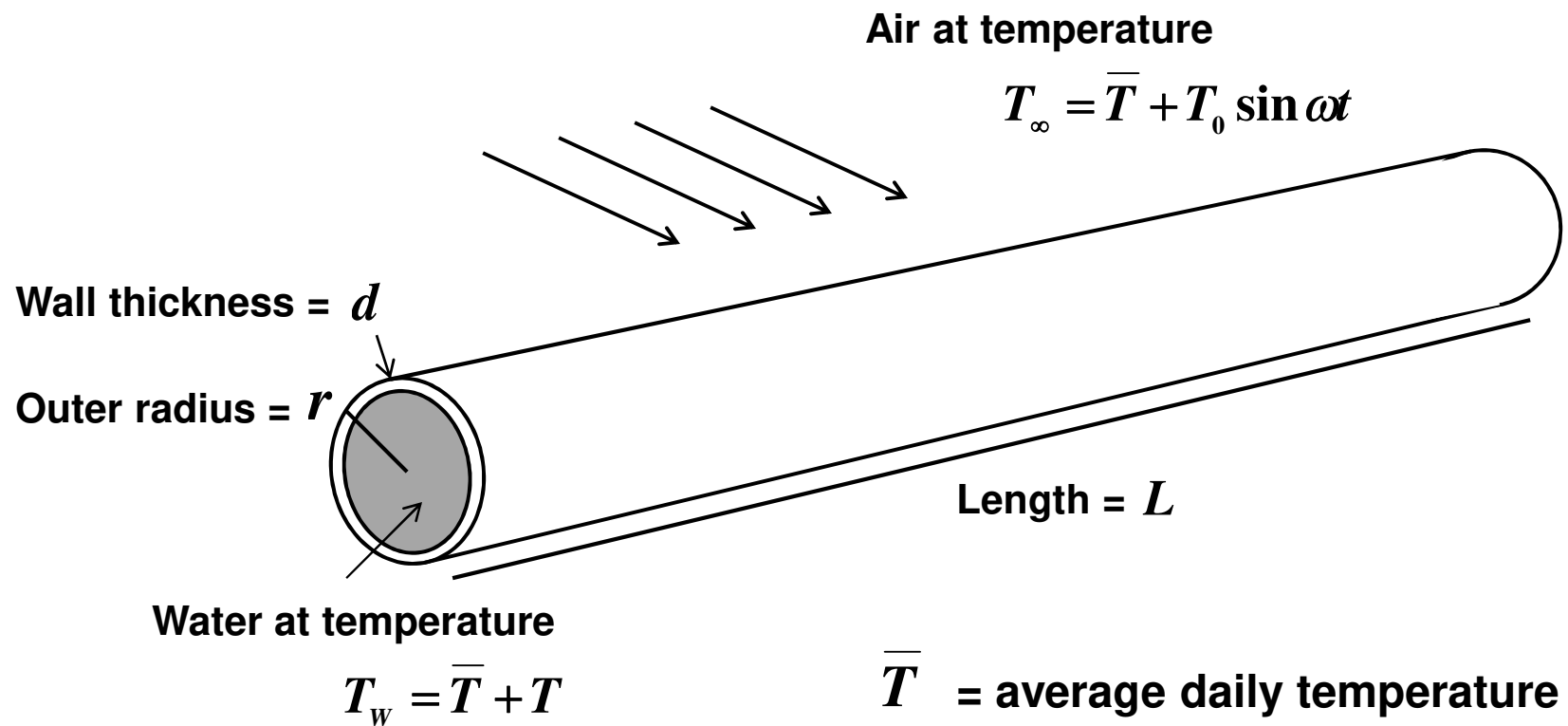
$$\frac{dy}{dt} + \left(\frac{1}{t}\right)y = 0$$

$$y(1) = c$$

First Order Linear Differential Equations

***Example: Exposed water pipe in cyclical
ambient temperature***

Exposed Water Pipe in Cyclical Ambient Temperature



Exposed Water Pipe in Cyclical Ambient Temperature (2)

Assuming pipe wall is thin and made of material that is a good heat conductor, by Newton's law of cooling, the heat transferred from air to water is

$$q = hA(T_{\infty} - T_w)$$

where

A = Exposed surface area of the pipe

h = Convection coefficient

Exposed Water Pipe in Cyclical Ambient Temperature (3)

The thermal energy stored in the water is

$$***E = mcT_w***$$

where

m = mass of the water

c = specific heat of water

Exposed Water Pipe in Cyclical Ambient Temperature (4)

Key physical principle:

$$\frac{dE}{dt} = q$$

which leads to

$$mc \frac{d}{dt} (\bar{T} + T) = hA \left((\bar{T} + T_0 \sin \omega t) - (\bar{T} + T) \right)$$

$$mc \frac{dT}{dt} + (hA)T = (hA)T_0 \sin \omega t$$

$$\frac{dT}{dt} + \lambda T = \lambda T_0 \sin \omega t \quad \text{where} \quad \lambda = \frac{hA}{mc}$$

Exposed Water Pipe in Cyclical Ambient Temperature (5)

How do we solve

$$\frac{dT}{dt} + \lambda T = \lambda T_0 \sin \omega t \quad ? \quad \text{Equation 2}$$

Let's be more inclusive and ask how do we solve the general linear first order nonhomogeneous equation

$$\begin{aligned} \frac{dy}{dt} + p(t)y &= g(t) & \text{Equation 3} \\ y(t_0) &= y_0 \end{aligned}$$

General Solution to Nonhomogeneous Linear First Order ODEs

We begin by searching for an integrating factor $\mu(t)$ that, when multiplied into the equation, turns the left-hand side into

$$\frac{d}{dt}(\mu(t)y)$$

Multiplying Equation 3 by $\mu(t)$:

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t)$$

Search for $\mu(t)$ such that left hand side is

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \frac{d}{dt}(\mu(t)y)$$

General Solution to Nonhomogeneous Linear First Order ODEs (2)

We must have

$$\mu(t) \frac{dy}{dt} + \mu(t) p(t) y = \frac{d}{dt} (\mu(t) y) = \mu(t) \frac{dy}{dt} + \frac{d\mu}{dt} y$$

which means that

$$\frac{d\mu}{dt} = p(t) \mu(t)$$

$$\frac{1}{\mu(t)} \frac{d\mu}{dt} = p(t)$$

General Solution to Nonhomogeneous Linear First Order ODEs (3)

$$\frac{d}{dt}(\ln \mu(t)) = p(t)$$

$$\ln(\mu(t) / \mu(t_0)) = \int_{t_0}^t p(u) du$$

$$\mu(t) = \mu(t_0) \exp \int_{t_0}^t p(u) du \quad \text{Equation 4}$$

This is the desired integrating factor.

But we can simplify the form.

General Solution to Nonhomogeneous Linear First Order ODEs (4)

We do not know what value to assign to $\mu(t_0)$ but it turns out not to matter. (The value cancels out.) So we set

$$\mu(t_0) = 1$$

It also suffices to use the indefinite integral form:

$$\mu(t) = \exp \int^t p(u) du \quad \text{Equation 5}$$

You should remember, or be able to derive, Equation 5

Exposed Water Pipe in Cyclical Ambient Temperature (6)

For the water pipe temperature problem (Equation 2):

$$\frac{dy}{dt} + p(t)y = g(t)$$

becomes

$$\frac{dT}{dt} + \lambda T = \lambda T_0 \sin \omega t$$

so

$$p(t) = \lambda$$

$$\mu(t) = \exp\left(\int^t p(u)du\right) = \exp\left(\int^t \lambda du\right) = e^{\lambda t}$$

Exposed Water Pipe in Cyclical Ambient Temperature (7)

Applying the integration factor to Equation 2:

$$e^{\lambda t} \left(\frac{dT}{dt} + \lambda T \right) = e^{\lambda t} (\lambda T_0 \sin \omega t)$$

$$\frac{d}{dt} (e^{\lambda t} T) = \lambda T_0 e^{\lambda t} \sin \omega t$$

**Now the value of the integration factor becomes clear:
We can solve the problem with an integration:**

$$e^{\lambda t} T(t) - T(0) = \lambda T_0 \int_0^t e^{\lambda \tau} \sin \omega \tau d\tau$$

$$T(t) = T(0)e^{-\lambda t} + e^{-\lambda t} \int_0^t e^{\lambda \tau} \sin \omega \tau d\tau$$

Exposed Water Pipe in Cyclical Ambient Temperature (9)

After performing the integral we have

$$T(t) = T_I e^{-\lambda t} + \left(\frac{\lambda}{\lambda^2 + \omega^2} \right) T_0 (\lambda \sin(\omega t) - \omega \cos(\omega t) + \omega e^{-\lambda t})$$

where

$$T_I = T(0) = T_w(0) - \bar{T}$$

Inhomogeneous First Order Linear ODEs: In-class Problems

$$\frac{dy}{dt} + \left(\frac{2}{t}\right)y = 4$$

$$y(1) = 2$$

$$\frac{dy}{dt} + 4\left(\frac{e^{4t} - e^{-4t}}{e^{4t} + e^{-4t}}\right)y = e^{3t}$$

$$y(0) = 6$$

$$\frac{dy}{dt} - (\tan t)y = \sec t$$

$$y(0) = 0$$

Homework Assignment 2

In text:

Read: Chapter 2

Work: On course website: Homework Assignment #2 Problems

Solutions for Homework Assignment #2 Problems will be provided on course website on (date)

Always read over the day's lecture notes and be sure you understand them.